# MONTHLY NOTICES

OF THE

# ROYAL ASTRONOMICAL SOCIETY,

CONTAINING

PAPERS, ABSTRACTS OF PAPERS, AND

REPORTS OF THE PROCEEDINGS

OF THE SOCIETY

*FROM NOVEMBER* 1925 *TO NOVEMBER* 1926.

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# MONTHLY NOTICES

OF THE

# ROYAL ASTRONOMICAL SOCIETY.

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Dr. J. H. JEANS, PRESIDENT, in the Chair.

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Robert Morrison Abraham, Luthers, Sewardstone, London, E. 4;
William Reginald Dean, M.A., Fellow of Trinity College, Cambridge, and Assistant Professor of Mathematics, Imperial College of Science and Technology, South Kensington, London, S.W. 7;
Joseph Cecil Maby, Marl Hill, near Cheltenham, Gloucestershire; and
Edward Wilfred Taylor, 10 Telford Terrace, York,

were balloted for and duly elected Fellows of the Society.

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The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Dr. E. S. Beaven, Eastway, Warminster (proposed by F. J. M. Stratton);
Thomas MacFarland Cherry, Ph.D., Fellow of Trinity College, Cambridge (proposed by W. M. Smart);
Clarendon Ions, 35 Southeast First Avenue, Miami, Florida, U.S.A. (proposed by J. Stokley);
Takehiko Matukuma, Assistant Professor of Astronomy, Tôhoku Imperial University, Sendai, Japan (proposed by W. M. Smart);
Maurice Rupert Meehan, 6 Bridge Street, Evesham, Worcestershire (proposed by W. E. Cooper);
Mrs. Dorothy Klumpke-Roberts (Mrs. Isaac Roberts), Château de By-par-Thomery, Seine et Marne, France (proposed by H. Knox-Shaw);
Clifford Somes Saxton, M.A., Whatley House, near Frome, Somerset (proposed by B. M. Peek); and
Francis Joseph Venning, Milton House, Arnison Road, East Molesey, Surrey (proposed by H. H. Waters).

Two hundred and four presents were announced as having been received since the last Ordinary Meeting, including, amongst others :—

A. Björnbo, Thabits Werk über den Transversalensatz . . . herausgegeben . . . von Dr. Bürger und Dr. Kohl.
H. B. Goodwin, A Companion to the Azimuth Tables.
Helsingfors Observatory, Astrographic Catalogue, tome i., fasc. 2.
Tokyo Imperial University, Anniversary Volume dedicated to Hantaro Nagaoka.
Rome, Specola Vaticana, Astrographic Catalogue, vol. 7 (+58°).
San Fernando Observatory, Carte du ciel, Zone −3° (10 charts).
F. J. M. Stratton, Astronomical Physics.
Sydney Observatory, Astrographic Catalogue, Sydney Section, Dec. −51° to −65°, vols. 1 and 2.
Observatoire Royal de Belgique, Carte du ciel, Zones +33°, +35°, +37°, +39° (20 charts).
Warsaw Observatory, Publications, vol. 1.
Washington, Navy Dept., Astronomical Papers, vol. 10, pt. 1, Positions and Motions of 1504 Standard Stars for the Equinox 1925·0.
Three Transparencies of Spectra of Sun and Moon, taken by Meldola and Lockyer (presented by Mr. Darling).

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*Electrostatic Forces in a Star and the Deviation from the Laws of a Perfect Gas.* By A. S. Eddington, F.R.S.

1. Until recently it was believed that the material of a star would cease to follow the laws of a perfect gas when its density approached that of water. This expectation was based on analogy with terrestrial gases. It is now realised that owing to the extremely small size of stellar ions compared with terrestrial atoms the analogy breaks down, and we do not expect deviations in the stars on account of atomic volume until enormous densities are reached. On the other hand, a possible source of deviation from the gas-laws exists in the ionised stellar gas, but not in terrestrial gases, viz. the electrostatic attractions and repulsions of the electrons and ions ; and it is important to determine how much effect this will have. The electrostatic effect and the atomic volume effect are quite dissimilar. I have shown in a former paper that the percentage correction to the pressure due to the electrostatic effect does not increase with the density of the star ; it depends only on the mass, being greatest in stars of small mass.*

A formula for this electrostatic effect has been developed by S. Rosseland † on the basis of a theory due to Debye and Hückel. Numerical calculations have been published by Fowler and Guggenheim,‡

* *Monthly Notices*, **84**, 320. Density has a very small indirect effect by changing the degree of ionisation of the material.

† *Monthly Notices*, **84**, 720. ‡ *Ibid.*, **85**, No. 9.

who used the same formula. Although I believe the theory of Debye and Hückel to be correct in principle, the approximations used in the calculations do not seem to me to be valid in the stellar problem, and the numerical results of Fowler and Guggenheim are probably about three times too large.* The deviations originally found by Fowler and Guggenheim were large enough to be of considerable astronomical importance ; as now reduced they reach a barely observable limit.

We give first the argument which it is here proposed to amend.

2. Consider material composed of positive ions of charge $Ze$ with the necessary number of free electrons to make it neutral. Let $\sigma_0$ be the number of ions and $s_0$ the number of free electrons per c.c. in regions where the electric potential is zero. Then at a place where there is a positive potential $\psi$, the average number of ions per c.c. will be

$$\sigma_0 e^{-Ze\psi/RT}$$

by Boltzmann's formula. The number of electrons per c.c. is similarly set equal to

$$s_0 e^{e\psi/RT}$$

although we shall see in § 6 that this is not justifiable. The total charge-density is thus

$$Ze \,.\, \sigma_0 e^{-Ze\psi/RT} - e \,.\, s_0 e^{e\psi/RT} \quad . \quad . \quad . \quad (1)$$

Consider the time-average of the potential and charge-density at a distance $r$ from a particular positive ion. If, following Rosseland, we expand the exponentials in (1) and neglect the square of $Ze\psi/RT$, the charge-density becomes

$$\rho = Ze\sigma_0(1 - Ze\psi/RT) - es_0(1 + e\psi/RT) \quad . \quad . \quad (2)$$

or since approximately $s_0 = Z\sigma_0$,

$$\rho = -Z(Z+1)\sigma_0 e^2\psi/RT = -\nabla^2\psi/4\pi \,. \quad . \quad . \quad (3)$$

by Poisson's equation. Since the time-average of the field is symmetrical about the positive ion, we take the symmetrical solution of (3), viz.

$$\psi = \frac{Ce^{-yr}}{r} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$y^2 = \frac{4\pi Z(Z+1)\sigma_0 e^2}{RT} \quad . \quad . \quad . \quad . \quad (5)$$

The charge within a sphere of radius $r$ is then

$$e' = -r^2\frac{d\psi}{dr} = C(1+yr)e^{-yr} \quad . \quad . \quad . \quad (6)$$

This is itself responsible for a potential at $r$,

$$\psi' = \frac{C(1+yr)e^{-yr}}{r} \quad . \quad . \quad . \quad . \quad (7)$$

---

* Mr. Fowler agrees with me that a correction on the lines here indicated is required.

leaving a potential due to the material outside $r$,

$$\psi-\psi'=-Cye^{-yr} \quad . \quad . \quad . \quad . \quad (8)$$

$$=-\frac{e'y}{1+yr} \quad . \quad . \quad . \quad . \quad (9)$$

Rosseland proceeds to calculate the electrostatic energy of the ion due to the surrounding charges by taking $r$ infinitely small. Then $e'$ becomes $Ze$, the extraneous potential $\psi-\psi'$ becomes $-Zey$, and the electrostatic energy of the ion $Ze$ is

$$-Z^2e^2y \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Evidently for an electron the energy is $-e^2y$ ; and the whole electrostatic energy per c.c. is thus

$$\begin{aligned} E&=-\tfrac{1}{2}\sigma_0Z^2e^2y-\tfrac{1}{2}s_0e^2y \\ &=-\tfrac{1}{2}Z(Z+1)\sigma_0e^2y \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

The factor $\frac{1}{2}$ appears because the mutual energy of each pair of charges would be reckoned twice over—once for each charge of the pair.

This is the formula which is also used by Fowler and Guggenheim. The correction to the pressure follows immediately from (11), since for inverse square forces the potential energy is equal to the virial.

3. The negative electrostatic energy arises from the circumstance that each ion concentrates negative electrons near it and repels other positive ions, so that on the average it surrounds itself with a negative charge distribution. Reference to (2) and (3) shows that the concentration of the electrons merely turns Z into Z+1 in the formulæ, so that it is not of much importance. In the main E comes from the repulsion of other positive ions.

In the approximation (2) the number of positive ions at a distance $r$ from the selected ion has been set equal to $\sigma_0(1-Ze\psi/RT)$, which becomes negative when $\psi$ is large and $r$ small. Actually the greater part of E in (11) is traceable to this *minus quantity* of ions, existing according to the approximate formula, but nonsensical in reality. Clearly it will be a better approximation (though still imperfect) to take the density of the ions zero when $Ze\psi/RT>1$ instead of negative.

We shall here make this change in order to remove the most glaring failure of the approximation in § 2. There is no doubt room for other criticism and amendment. The present paper seeks only to show that a large part of the electrostatic deviation from the gas-laws given by previous investigations is spurious.

4. Let $r_0$ be a distance from the ion such that

$$Ze\psi_0/RT=1 \quad . \quad . \quad . \quad . \quad (12)$$

where $\psi_0$ is the potential at $r_0$. Then the formulæ of § 2 are taken to be valid for $r>r_0$. Within $r_0$ we suppose that there are no positive ions (as an improvement on the previous assumption that there is a negative number of ions) ; we also suppose that the negative electrons are uniformly distributed with density $s_0=Z\sigma_0$, since the previous work showed that their concentration was not responsible for any important part of E.

Hence the charge $e_0'$ within $r_0$ is

$$Ze(1-\tfrac{4}{3}\pi r_0^3\sigma_0)=C(1+yr_0)e^{-yr_0} \quad . \quad . \quad . \quad (13)$$

by (6). Also by (12) and (4)

$$\frac{Ze}{RT}\frac{Ce^{-yr_0}}{r_0}=1 \quad . \quad . \quad . \quad . \quad (14)$$

Writing

$$z=yr_0, \quad a=Z^2e^2y/RT, \quad b=y^3/\tfrac{4}{3}\pi\sigma_0 \quad . \quad . \quad (15)$$

and eliminating C between (14) and (13), we obtain

$$\frac{z^3}{b}+\frac{z^2}{a}+\frac{z}{a}=1 \quad . \quad . \quad . \quad . \quad (16)$$

so that $z$ can be determined.

The charge within $r_0$ is then

$$e_0'=Ze(1-z^3/b)=Ze(z+z^2)/a \quad . \quad . \quad . \quad (17)$$

The potential at $r_0$ is

$$+e_0'/r_0$$

due to material inside $r_0$, and

$$-e_0'z/(1+z)r_0$$

due to material outside $r_0$, the latter result being obtained from (9).

The potential at the centre (apart from that due to the ion itself) is

$$-\tfrac{3}{2}(Ze-e_0')/r_0$$

due to the uniform distribution of electrons (total charge $=e_0'-Ze$) in a sphere of radius $r_0$, and

$$-e_0'z/(1+z)r_0$$

due to material outside $r_0$.

5. For an example, we take one of the stars considered by Fowler and Guggenheim, the material being Fe, the density and temperature being

$$\rho=156, \quad T=26{\cdot}36 \,.\, 10^6.$$

Then

$$\sigma_0=\rho/AH=1{\cdot}68 \,.\, 10^{24}.$$

With Z=24 (iron nucleus+two electrons), we obtain by (5)

$$y=8{\cdot}93 \,.\, 10^8.$$

Hence by (15) and (16)

$$a=32{\cdot}4, \quad b=101{\cdot}3, \quad z=3{\cdot}65, \quad r_0=4{\cdot}1 \,.\, 10^{-9} \text{ cm.}$$

Notice that $r_0$ is much larger than the radius of the ion (about $2 \,.\, 10^{-10}$ cm.) so that the finite size of the ion does not give rise to trouble. Then we find by (17)

$$e_0'=0{\cdot}520Ze$$

that is to say, there are 0·480Z free electrons within the sphere of radius $r_0$. The potential at $r_0$ is

$$+0{\cdot}520Ze/r_0$$

due to material inside $r_0$ (including the ion itself), and

$$-0{\cdot}408Ze/r_0$$

due to material outside $r_0$.

The potential energy of the ion due to matter outside $r_0$ is

$$-0{\cdot}408Z^2e^2/r_0=-0{\cdot}112Z^2e^2y$$

and that due to the $0{\cdot}480Z$ free electrons in a sphere of radius $r_0$

$$-\tfrac{3}{2}\times 0{\cdot}480Z^2e^2/r_0=-0{\cdot}198Z^2e^2y$$

giving a total of

$$-0{\cdot}31Z^2e^2y.$$

Comparing with (10), we see that the electrostatic energy is only 31 per cent. of that calculated by Fowler and Guggenheim. For the star here considered they calculate the gas pressure to be 21·9 per cent. less than the perfect gas value; our correction makes it 6·8 per cent. less than the perfect gas value.

6. In (2) the density of the free electrons near the ion was set equal to $s_0(1+e\psi/\mathrm{RT})$. Since in Rosseland's treatment the contribution to E from this concentration is trivial, we have not paid further attention to it. It might perhaps be urged that since this expression was introduced as an approximation for $s_0e^{e\psi/\mathrm{RT}}$ it greatly underestimates the density of the electrons near the ion where $\psi$ is large, and that if the true expression were used we should increase E, restoring it towards Fowler and Guggenheim's values. The argument, in short, would be that the approximations for ions and electrons are both pretty bad, but we do more harm than good by improving the approximation for one without improving it for the other.

The fact is, however, that the expression $s_0e^{e\psi/\mathrm{RT}}$ for the density of the negative electrons is not correct. By the usual Maxwell-Boltzmann formula the density of free electrons in a region of potential $\psi$ is

$$s=\int_0^\infty\left(\frac{m}{2\pi\mathrm{RT}}\right)^{\frac{3}{2}}s_0e^{-(\frac{1}{2}m\mathrm{V}^2-e\psi)/\mathrm{RT}}\cdot 4\pi\mathrm{V}^2d\mathrm{V}\quad(\psi<0)\quad .\quad (18)$$

$$s=\int_{\mathrm{V}_0}^\infty\left(\frac{m}{2\pi\mathrm{RT}}\right)^{\frac{3}{2}}s_0e^{-(\frac{1}{2}m\mathrm{V}^2-e\psi)/\mathrm{RT}}\cdot 4\pi\mathrm{V}^2d\mathrm{V}\quad(\psi>0)\quad .\quad (19)$$

where

$$\tfrac{1}{2}m\mathrm{V}_0{}^2=e\psi.$$

The former expression gives

$$s=s_0e^{e\psi/\mathrm{RT}}\quad(\psi<0)\quad .\quad .\quad .\quad .\quad (20)$$

The latter reduces to

$$s=\frac{4}{\sqrt{\pi}}s_0e^{x^2}\int_x^\infty e^{-t^2}t^2dt\quad(\psi>0)\quad .\quad .\quad .\quad (21)$$

$$=\frac{2}{\sqrt{\pi}}s_0\left\{x+e^{x^2}\int_x^\infty e^{-t^2}dt\right\}$$

where

$$x^2=e\psi/\mathrm{RT}.$$

The necessity for the lower limit to the integral in (19) is due to the fact that an electron for which $\frac{1}{2}mV^2 - e\psi$ is negative is not free. For small negative energy the captured electrons are given correctly by continuing the integral below $V_0$, but ultimately the Maxwell-Boltzmann expression has to be replaced by the quantum laws.

The following table shows the course of $s/s_0$ according to (21) as compared with the Boltzmann formula $e^{e\psi/RT}$ (which applies only to negative values of $\psi$) and Rosseland's approximation $1+e\psi/RT$ :—

| $e\psi/RT$. | $s/s_0$. | $e^{e\psi/RT}$. | $1+e\psi/RT$. |
|---|---|---|---|
| 0 | 1·00 | 1·00 | 1·0 |
| $\frac{1}{2}$ | 1·32 | 1·65 | 1·5 |
| 1 | 1·56 | 2·72 | 2·0 |
| 2 | 1·93 | 7·39 | 3·0 |
| 3 | 2·24 | 20·09 | 4·0 |
| 4 | 2·51 | 54·60 | 5·0 |

It will be seen that the concentration of electrons was by no means underestimated in the original discussion; it was considerably exaggerated. At $r_0$ we have (for iron) $e\psi/RT = \frac{1}{2\cdot 4}$, so that it is not until $\psi$ reaches $50\psi_0$ that the density of electrons is doubled. Close to the nucleus $s$ tends to infinity like $r^{-\frac{1}{2}}$, so that the number in the neighbourhood of the nucleus is finite.

7. On the basis of the approximation here adopted, I have calculated the following corrections $\Delta m$ to the absolute magnitude of a star of mass M, *i.e.* $-\Delta m$ is the increase of brightness as compared with a star of the same mass obeying the gas-laws. The material is taken to be iron:

| M. | $-\Delta m$. m | $-\Delta p_G/p_G$. |
|---|---|---|
| 1·04 | 0·21 | ·033 |
| 0·70 | 0·29 | ·043 |
| 0·32 | 0·50 | ·068 |
| 0·205 | 0·68 | ·089 |

The last column shows that, for example, in a star of mass 0·205 the gas pressure is reduced by 8·9 per cent. on account of the electrostatic forces.

For comparison I give the results deduced from Fowler and Guggenheim's approximation:

| M. | $-\Delta m$. m | $-\Delta p_G/p_G$. |
|---|---|---|
| 1·99 | 0·25 | ·050 |
| 1·14 | 0·44 | ·072 |
| 0·63 | 0·85 | ·112 |
| 0·42 | 1·13 | ·149 |
| 0·248 | 1·78 | ·219 |
| 0·120 | 3·07 | ·344 |

If the first set of corrections is applied to my mass-luminosity curve, the curve is raised slightly on the left of the diagram so as to improve a little the agreement of theory and observation; but the observed points for the small stars still tend to be rather above the curve. The main conclusion is that deviations from the laws of a perfect gas will not affect stellar luminosity by more than about half a magnitude. The corrections here found ought perhaps to be further reduced, since the electrostatic effect is still exaggerated by the formulæ adopted; only the most conspicuous fault has been removed.

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*The Equilibrium of the Calcium Chromosphere.* (*Second Paper.*)
By Professor E. A. Milne.

1. *Outline of the Paper.*—In two recent papers * I have discussed the equilibrium of the high-level calcium atoms in the sun on the hypothesis that they are supported against gravity by radiation pressure, and I have shown how the average life $\tau$ of an ionised calcium atom in the $1\pi$ state may be deduced from observations of the residual intensity at the centres of the H and K absorption lines in the solar spectrum. In order to present the argument as simply as possible, certain simplifying assumptions were introduced. Amongst others, it was assumed (1) that the "statistical weights" of the normal and excited states were equal; (2) that the $1\pi_1$ and $1\pi_2$ levels could be treated as forming a single $1\pi$ level; (3) that the $1\sigma$ (normal) and $1\pi$ (excited) levels were the only ones which the series electron could occupy. In other words, it was assumed that the $Ca^+$ atom possessed two, and only two, stationary states of equal weight. In the present paper, substantially all these restrictions are removed. (1) Account is taken of the unequal weights of the $1\sigma$ and combined $1\pi$ levels. (2) Account is taken of the duplicity of the $1\pi$ level; it is shown what is the correct mean value of the residual intensities in the H and K lines to use in estimating $\tau$, and it is shown how to calculate these residual intensities separately. (3) Account is taken of the metastable $1\delta$ levels of the $Ca^+$ atom; it is shown in what way these modify the previous estimate of $\tau$, and how the residual intensities in the resultant $(1\delta-1\pi)$ lines ($\lambda\lambda$8498, 8542, 8662) may be calculated, and it is deduced that these lines should appear, as emission lines, at all heights in the chromosphere at which the H and K lines appear. It will perhaps be simplest to explain the results in detail first, leaving the mathematics to the later part of the paper.

(1) Let $q_1$ be the statistical weight of the $1\sigma$ state, $q_2$ that of the combined $1\pi$ states. We continue for the present to regard the $1\pi$ states as forming a single state, giving rise to a single line, of frequency

* "An Astrophysical Determination of the Average Life of an Excited Calcium Atom," *M.N.*, **84**, 354, 1924; "The Equilibrium of the Calcium Chromosphere," *M.N.*, **85**, 111, 1924. The latter will be referred to as Paper I.

$\nu$ and residual intensity $F_\nu$. Let $g$ be the value of gravity, $m$ the mass of the atom, $c$ the velocity of light. It is shown in Section I. that for the high-level atoms to be just supported by radiation pressure against gravity, $F_\nu$ must be given approximately by the relation

$$F_\nu = \frac{q_1}{q_2} \cdot \frac{8\nu^2 \tau m g}{c} \quad . \quad . \quad . \quad . \quad (1)$$

This reduces to the formula given previously on putting $q_1 = q_2$.

The combined weight * of the states of azimuthal quantum number $k$, in a system of maximum multiplicity $2i$, is $2i(2k-1)$. For $Ca^+$ we have $2i=2$, and $k=1$, 2 for the $\sigma$ and $\pi$ levels respectively. Hence $q_1=2$, $q_2=6$, and $q_1/q_2=\frac{1}{3}$. Thus for given $\tau$ the effect of taking account of the weights is to reduce the residual intensity by a factor of $\frac{1}{3}$ for the H and K lines. The physical reason for this is easily seen. In effect, when $q_2/q_1=3$ the atom has 3 times as many ways of making the transition as when $q_2/q_1=1$. It is thus exposed to 3 times the radiation pressure, and therefore the residual intensity in the line need have only $\frac{1}{3}$ of its value in order to support the atoms against gravity.

Actually, however, we use the observed value of $F_\nu$ in (1) and deduce a value for $\tau$. For given $F_\nu$, the determined value of $\tau$ now comes out $q_2/q_1$ times bigger than before. Taking account of the weights, the value $\tau = 0{\cdot}6 \times 10^{-8}$ sec. found previously must now be multiplied by 3, giving $\tau = 1{\cdot}8 \times 10^{-8}$ sec. This brings it into closer agreement with the observed values for hydrogen.†

It should be observed that the factor $q_2/q_1$ has no counterpart in the classical theory of radiation.

The formulæ for the density-distribution, etc., require consequential alterations to take account of the $q$'s. It appears that the main characteristics of the equilibrium are unaltered.

(2) We have calculated as though the $1\pi$ level were single, of weight $q_2$. Actually it is double, consisting of two levels, $1\pi_1$ and $1\pi_2$, of weights say $q_2'$ and $q_2''$. We have, of course,

$$q_2' + q_2'' = q_2.$$

In the notation of Russell and Saunders, the $1\pi_1$ and $1\pi_2$ levels are denoted by $1^2P_2$ and $1^2P_1$; the inner quantum numbers have the values 2 and 1 respectively, and the system being one of even multiplicity the statistical weights are $q_2' = 4$, $q_2'' = 2$.

Let $F_{\nu'}$, $F_{\nu''}$ be the residual intensities in the corresponding lines. It is shown in Section II. that the high-level atoms will be just supported by radiation pressure against gravity, provided

$$q_2' F_{\nu'} + q_2'' F_{\nu''} = \frac{8\nu^2 \tau m g}{c} q_1 \quad . \quad . \quad . \quad (1a)$$

---

* Landé, *Zeit. für Phys.*, **15**, 189, 1923. In Landé's notation the combined weights are ${}_4RK$, and $R=i$, $K=k-\frac{1}{2}$.

† *Cf.* the table given by Miss Payne, "Stellar Atmospheres" (*Harvard Monographs*, No. 1), p. 22.

approximately, where on the right-hand side we have taken a mean value for $\nu$. It will be observed that we have used a single value $\tau$ for the life in each of the excited states. That these are equal follows from the summation rules of Burger and Dorgelo,* as shown by Ornstein and Burger.†

The consideration of mechanical equilibrium alone thus provides only one relation between the two unknown residual intensities $F_{\nu'}$ and $F_{\nu''}$. We have simply expressed that between them the two residual intensities must be sufficient to support the atoms against gravity. If the two residual intensities were equal, (1*a*) would reduce to (1), and our previous calculation would give $\tau$ correctly.

But a complete theory should be capable of determining $F_{\nu'}$ and $F_{\nu''}$ separately from the atomic constants. According to the summation rules of Burger and Dorgelo, the intensities of the K and H lines *in emission* should be in the ratio of the weights of the corresponding initial states, namely 4 : 2 or 2 : 1.

Such a ratio is observed for the corresponding Na lines. Measures of the ratio for K and H in the laboratory have been made by Dorgelo.‡ The lines were more nearly equal than in the ratio 2 : 1, but this could be attributed to the effects of self-reversal. As the tendency to self-reversal was reduced, the ratio approached more nearly 2 : 1, and Dorgelo expressed the opinion that if self-reversal were completely eliminated the ratio would in fact be found to have its theoretical value 2 : 1. The same ratio should apply to the intensities of the absorption lines formed by a thin layer of gas—the K should be twice as strong in absorption as the H line, *i.e.* its residual intensity should be $\frac{1}{2}$ that of the H line. We cannot apply these results as they stand to a stratified mass of gas such as the solar atmosphere, in which the gas at different levels is in different physical conditions. But it is possible to integrate through the different strata, and the detailed calculations of Section II. show that approximately the same result should hold, namely, that the residual intensity in the K line should be $\frac{1}{2}$ that in the H line, on the hypotheses adopted. In algebraic language, we ought to have

$$q_2'F_{\nu'}=q_2''F_{\nu''},$$

so that if we write $\overline{F}_\nu$ for the mean intensity, defined by

$$q_2\overline{F}_\nu=q_2'F_{\nu'}+q_2''F_{\nu''},$$

then

$$F_{\nu'}=\tfrac{1}{2}\left(1+\frac{q_2''}{q_2'}\right)\overline{F}_\nu, \quad F_{\nu''}=\tfrac{1}{2}\left(1+\frac{q_2'}{q_2''}\right)\overline{F}_\nu,$$

and

$$\frac{1}{F_\nu}+\frac{1}{F_{\nu''}}=\frac{2}{\overline{F}_\nu}.$$

$\overline{F}_\nu$ is the residual intensity that would be given by the simple formula (1).

According to these, the residual intensity in the K line should be about 0·75 magnitude fainter than in the H line. Schwarzschild's

* *Zeit. für Phys.*, **22**, 258, 1924. † *Ibid.*, **24**, 41, 1924.
‡ *Ibid.*, **31**, 827, 1925.

measures * of the contours of the H and K lines, which are the only ones available, show the residual intensity in K to be only 0·25 magnitude fainter than in H. This resembles the laboratory results in giving the ratio K : H more nearly unity than its theoretical value, and is doubtless to be attributed to the same phenomenon, self-reversal. It is well known, in fact, that the structure at the centres of the K and H lines in the solar spectrum is often very complicated. There is also the complication due to the presence of $H_\epsilon$. Further, an accurate determination of the residual central intensities from Schwarzschild's curves is not easy ; the contours are of a marked V-form, the shoulders sloping at a rate of about one magnitude per 2 Å. Clearly, accurate measures are wanted with large dispersion.

Since the observed $F_{\nu'}$ and $F_{\nu''}$ are approximately equal, no correction is required to our deduced value for $\tau$ for the doublet character of the $1\pi$ level, but one may be required in future if observation ultimately confirms the 2 : 1 ratio.

We may point out here that the accurate determinations of the relative residual intensities in the components of multiplets in stellar spectra would probably give results of great value. The values calculated by recent investigators † from the quantum theory agree astonishingly well with the relative intensities observed in the laboratory, and it would be a matter of great interest to know if similar intensity relations hold for absorption lines in stellar spectra.

(3) Between the $1\sigma$ and $1\pi$ levels of the $Ca^+$ atom there occur the $1\delta$ levels. The actual values in wave numbers, taken from A. Fowler's "Report," are as follows :—‡

| | | |
|---|---|---|
| $(1^2S_1)$ | $1\sigma$ | 95739·70. |
| $(1^2P_2,\ 1^2P_1)$ | $1\pi_1,\ 1\pi_2$ | 70325·29, 70548·14. |
| $(1^2D_3,\ 1^2D_2)$ | $1\delta,\ 1\delta'$ | 82028·73, 82089·52. |

This set of levels gives rise to the following lines :—

| | | | |
|---|---|---|---|
| $(1^2S_1-1^2P_2)$ | $1\sigma-1\pi_1$ | $\lambda$3933 | Principal doublet (K, H). |
| $(1^2S_1-1^2P_1)$ | $1\sigma-1\pi_2$ | 3968 | |
| $(1^2D_2-1^2P_2)$ | $1\delta'-1\pi_1$ | 8498 | Diffuse doublet with satellite (X). |
| $(1^2D_3-1^2P_2)$ | $1\delta-1\pi_1$ | 8542 | |
| $(1^2D_2-1^2P_1)$ | $1\delta'-1\pi_2$ | 8662 | |

Transitions of the type $(1\sigma-1\delta)$ do not occur because they would involve a change of azimuthal quantum equal to 2, which is forbidden by the correspondence principle. Similarly (though this is unimportant for our present purpose) the transition $(1\delta-1\pi_2)$ is forbidden, as it would involve a change of inner quantum number equal to 2. Lines corresponding to the forbidden transitions are not observed. The $(1\sigma-1\pi)$ lines are, of course, the H and K lines of the solar spectrum ;

* *Berlin Sitz.*, 1914, p. 1183.

† *E.g.*, Kronig, *Zeit. für Phys.*, **31**, 885, 1925 ; H. N. Russell, *Proc. Nat. Acad. Sci.*, **11**, 314, 1925.

‡ The notation suggested by Russell and Saunders, *Astrophys. Journ.*, **61**, 39, 1925, is inserted in parentheses.

the ($1\delta - 1\pi$) lines form a well-known group (known as X) in the sun's infra-red spectrum. They are shown, for example, in Meggers' photographs.*

An atom in a $1\pi$ state can emit one of the H and K lines and pass to the $1\sigma$ state, or alternatively it can emit a member of the infra-red diffuse group and pass to a $1\delta$ state. An atom in a $1\delta$ state cannot pass directly to the $1\sigma$ state, though the $1\sigma$ state is one of lower energy, for this would involve a change of azimuthal quantum number equal to 2. Thus an atom in the $1\delta$ state must stay in that state † until it absorbs a quantum and passes to a state of higher energy; it cannot spontaneously pass to the $1\sigma$ state. The $1\delta$ states are thus metastable —they possess certain of the properties of the normal state. For our purpose the $1\delta$ states are in fact completely comparable with the $1\sigma$ state—it is as though the $Ca^+$ atom possessed additional normal states. This will have an effect similar to increasing the weight of the normal state.

The importance of this state of affairs for the high-level chromosphere is obvious. At every height at which the H and K lines are observed, $Ca^+$ atoms are oscillating between the $1\sigma$ and $1\pi$ states. But of the atoms which arrive in the $1\pi$ state in any interval of time, a fraction will pass to one of the $1\delta$ states, emitting a member of the X group. It follows that the lines of the diffuse doublet in the infra-red are to be expected at every height at which the bright H and K lines appear.

This inference does not depend on any theory of the equilibrium of the high-level atoms. From the occurrence of the H and K lines we at once infer the existence of atoms in the $1\pi$ state, and therefore of the occurrence of transition downwards to the $1\delta$ state. Thus a photo of the flash spectrum taken with a plate equally sensitive in the regions $\lambda$3950 and $\lambda$8580 should show arcs for the lines $\lambda\lambda$8498, 8542, 8662 comparable in length with those for H and K. Which should actually be the longer on the plate depends on the relative intensities, for one group may pass below the threshold intensity for the plate before the other. But there can be no stratification of the H, K radiation and the X-group radiation in different horizons in the chromosphere; the latter must be being emitted whenever the former is, and conversely.

This appears to be the explanation of the recent results of Curtis and Burns.‡ They obtained the infra-red flash spectrum at the eclipse of 1925 Jan. 24, and they report that a striking feature in the region $\lambda$6100–8800 is the strength of calcium. The three lines $\lambda\lambda$8498, 8542, 8662 are prominent features, giving long arcs, in a region in which the sensitivity was falling off rapidly. Their observations do not permit a comparison of the strengths and heights of these lines with H and K. The only longer arcs in the region mentioned were those

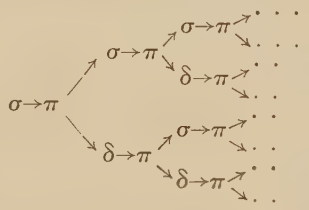
* *Astrophys. Journ.*, **47**, 1, 1918. According to the summation rules, the intensities of $\lambda\lambda$8498, 8542, 8662 should be in the ratio 1 : 9 : 5. In the present paper I have for brevity omitted the consideration of the calculation of their relative intensities in the Fraunhofer spectrum, but in analogy with the results of Section II. we may expect them to have the relative intensities just given. In Megger's photograph the line $\lambda$8542 is much the strongest.

† In the absence of collisions.

‡ *Pub. Allegheny Obs.*, **6**, 95, 1925.

for $H_\alpha$ ($\lambda$6563) and an oxygen line near $\lambda$7800, where the plate sensitivity must have been much stronger. There is nothing to suggest that the observations are incompatible with the theoretical prediction, that the X-group should be comparable in length with H and K. In any case, the relative prominence of the X-group at high levels is accounted for as an inevitable consequence of the existence of the metastable $1\delta$ states.

Thus theory and observation are in general agreement. Confining attention to these three states only,* the possible transition of a $Ca^+$ atom may be represented thus :




At any instant, at any height, there will be atoms present in all three states, $1\sigma$, $1\pi$, and $1\delta$, and for mechanical equilibrium against gravity we must seek the condition that the totality of permissible transitions provides the radiation pressure necessary for supporting the atoms. Each atom will spend a certain definite fraction of its time in the $1\sigma$ state, a certain fraction in the $1\pi$ state, and a certain fraction in the metastable $1\delta$ state.

Our first object is to calculate the residual intensities both in the H, K doublet and in the X-group. For convenience, we regard the $1\delta_1$ and $1\delta'$ levels as forming a single $1\delta$ level, and we continue to regard the $1\pi_1$ and $1\pi_2$ levels as forming a single $1\pi$ level, each of their appropriate weight.† This is equivalent to supposing the doublets combined into single lines of wave-lengths $\lambda$3950 and $\lambda$8580. We denote by $F_\nu$, $F_{\nu'}$ the residual intensities in these lines. Then it appears that the high-level atoms will be just supported against gravity provided

$$\nu^3\left(\frac{c}{8\nu^2\tau mg}-\frac{q_1/q_2}{F_\nu}\right)+\nu'^3\left(\frac{c}{8\nu'^2\tau' mg}-\frac{q_1'/q_2}{F_{\nu'}}\right)=0 \qquad . \qquad (2)$$

approximately,‡ where the prime (′) refers throughout to the metastable state.

Again we have two unknowns, $F_\nu$ and $F_{\nu'}$, but only one equation. The residual intensities of the two lines are not determinate from consideration of the high-level atom only. It is easy to see the physical reason for this. Equation (2) is the condition that the *total*

* The two $1\pi$ states are regarded as forming a single $1\pi$ state here, the two $1\delta$ states similarly. Transitions are, of course, possible between the $1\sigma$ state or $1\delta$ state and the $\alpha\pi$, $3\pi$ ... states, and between the $1\delta$ state and the $\varphi$ states. It would not be difficult to trace in a general way the effects of such transition, but an exact treatment would be very complicated.

† The $1\delta$, $1\delta'$ levels have weights 6, 4 respectively, total 10. As in (2), the intensities thus derived will be the weighted mean intensities for the multiplets.

‡ The exact form, equation (59) below, contains a small additional term.

radiation pressure derived from the two lines must be equal to the weights of the atom, but we have imposed no condition which decides the relative shares taken by the two lines in supporting the atom. If the gain of momentum due to gravity in each duration in the $1\sigma$ state were balanced by the succeeding $\nu$-absorption, and each duration in the $1\delta$ state by the succeeding $\nu'$-absorption, each term in (2) would be separately zero. But we have no right to assume this to be the case.

If $A_{2\to1}$, $A_{2\to1'}$ are the Einstein probability coefficients, and if we are looking at a thin layer of gas, the relative emission intensities will be in the ratio $h\nu' A_{2\to1'} : h\nu A_{2\to1}$, for this is the ratio of the numbers of quanta emitted multiplied by the ratio of the energies of the $\nu'$- and $\nu$-quanta. This, then, will be the ratio of the intensities at great heights in the flash. But it will not be the ratio of the residual intensities in the Fraunhofer spectrum. For here we are looking at a thick layer, and the residual intensity is the resultant of contributions derived from atoms at various optical depths. The bigger $A_{2\to1'}/A_{2\to1}$ is, the bigger is also the ratio of the atomic absorption coefficients. Thus we can only find the relative residual intensities by taking account of the atoms at all depths. It is now clear that any ratio whatever between $F_\nu$ and $F_{\nu'}$ is possible. For it depends on the intensities in $\nu$ and $\nu'$ incident on the atmosphere from below. By making the incident $\nu$ radiation very strong and the incident $\nu'$ radiation very weak, we could make $F_\nu$ strong and $F_{\nu'}$ weak.

The assumption which it is reasonable to make, is that at a certain optical depth the outward intensities have the photospheric values characteristic of some temperature $T_1$. We trace the emergent rays backwards until we arrive at a depth where one of them has the intensity of black $\nu$-radiation at temperature $T_1$; we then impose the condition that at this depth the $\nu'$-radiation must also have the intensity for temperature $T_1$. This provides a second equation between $F_\nu$ and $F_{\nu'}$, and the problem is solved. For $T_1$ we may adopt the sun's effective temperature.

Ideally the problem is that of the steady state of a slab of gas exposed to intensities $I_\nu(T_1)$ and $I_{\nu'}(T_1)$ at its lower boundary. It can be shown that separate radiative equilibria * exist in the frequencies $\nu$ and $\nu'$, and accordingly formulæ due in the first instance to Schuster connect $F_\nu$ and $F_{\nu'}$ with $I_\nu(T_1)$ and $I_{\nu'}(T_1)$. Connecting $I_\nu(T_1)$ and $I_{\nu'}(T_1)$ by Planck's law, we have the required second equation between $F_\nu$ and $F_{\nu'}$.

Our second equation involves, however, considerable uncertainties.

* It may be asked, cannot we have continuous conversion of $\nu$-radiation into $\nu'$-radiation, so that though there is an equilibrium of radiation as a whole there is not one in the separate frequencies? This will actually occur in the *setting up* of the steady state, but in the steady state itself the number of transitions $1\to2$ must equal the number $2\to1$, and similarly for $1'\to2$ and $2\to1'$. A paradoxical case occurs when the incident radiation $I_{\nu'}$ is taken to be zero; it might be thought that surely in this case there would be *some* emergent $\nu'$-radiation, obtained at the expense of the $\nu$-radiation. This is not so. Since there is no incident $\nu'$-radiation, once an atom is in the $1'$ state it stays there; thus all the atoms will rapidly go into the $1'$ state by the path $1\to2\to1'$. The gas is then completely transparent to $\nu$-radiation, which goes through with unimpaired intensity, and no further $\nu'$-radiation is originated.

For it assumes that we know enough about the structure of the atmosphere down to the photospheric level to do the calculations ; it requires a knowledge of $T_1$, and (as it turns out) it requires a knowledge of the ratio of the absorption coefficients for atoms in the states 1 and 1′. Fortunately it can be seen that for $Ca^+$, owing to $\nu'$ being so far in the infra-red, the term in $F_{\nu'}$ in (2) is relatively small. Neglecting it, we find that the correction due to the existence of the 1δ state amounts to replacing $\tau$ by $\tau/(1+\nu'\tau/\nu\tau')$ in the simple formula (1). For given $F_\nu$, it appears that the deduced value of $\tau$ is increased by a factor 1·46, giving $\tau = 1{\cdot}8\times 1{\cdot}46\times 10^{-8} = 2{\cdot}6\times 10^{-8}$ sec., assuming for definiteness that $\tau'=\tau$. But it should be noted that the true "average life in the excited state" * is now not $\tau$ but $\tau\tau'/(\tau+\tau')$, which if we take $\tau'=\tau$ is $\frac{1}{2}\tau$ or $1{\cdot}3\times 10^{-8}$ sec. The quantity $\tau$ is simply the reciprocal of $A_{2\to 1}$, $\tau'$ of $A_{2\to 1'}$, so that it is only in an extended sense that $\tau$ or $\tau'$ can be spoken of separately as "average lives."

From our second equation it is possible to calculate the intensity-ratio † in λ8580, given that in λ3950, if we know $\tau'/\tau$. Taking the intensity-ratio in H and K as 0·11 from observation, and putting $\tau'=\tau$, we find about 0·4 for the intensity-ratio for the diffuse doublet. As absorption lines, therefore, the infra-red lines should be nearly 4 times weaker than the H and K lines. No actual measures appear to exist for the infra-red $Ca^+$ lines. Comparison of actual measures with the theoretical value might furnish valuable information concerning the structure of the lower chromosphere and the ratio $\tau'/\tau$.

It appears further that the theoretical value of the residual intensity in the infra-red lines is about 10 times that in the H and K lines ; this is the product of the factor 0·4/0·11 for the intensity ratios and the factor 2·7 for the relative background intensities $I_{\nu'}(T_1)$ and $I_\nu(T_1)$. Consequently the infra-red line in the flash should be 10 times brighter than the H and K lines at low levels. This is at first sight a surprising result, but it is merely a consequence of the general principle that the fainter line in the Fraunhofer spectrum gives the brighter line in the flash.

## I. *Effect of* a priori *Weights.*

2. *The Fundamental Formulæ.*—We employ the notation of Paper I. When $q_1$, $q_2$ are the *a priori* weights of an atom in the states 1 and 2 respectively, Einstein's relations between the probability coefficients are

$$\frac{A_{2\to 1}}{B_{1\to 2}}=\frac{2h\nu^3}{c^2}\cdot\frac{q_1}{q_2}, \qquad \frac{B_{2\to 1}}{B_{1\to 2}}=\frac{q_1}{q_2}. \qquad (3), (4)$$

If $\alpha(\nu)$ is the atomic absorption coefficient, we find that the integral through the line is given as before by

$$\int \alpha(\nu)d\nu=\frac{B_{1\to 2}h\nu}{4\pi}.$$

* The total probability of leaving the state 2 being $(A_{2\to 1}+A_{2\to 1'})dt$, the average life is $(A_{2\to 1}+A_{2\to 1'})^{-1}$, which is $\tau\tau'/(\tau+\tau')$.

† Intensity ratio means (central residual intensity/adjacent continuous spectrum).

Introducing $\tau$, the mean life in the excited state, defined by

$$\tau = 1/A_{2\to 1}$$

we find

$$\int \alpha(\nu)d\nu = \frac{c^2}{8\pi\tau\nu^2}\cdot\frac{q_2}{q_1} \quad . \quad . \quad . \quad (5)$$

As explained in the previous paper, it is convenient to suppose that the line is sharply bounded, and of width $\Delta\nu$, and that within this range the absorption coefficient is constant and equal to $\alpha$. Relation (5) is then written

$$\alpha\Delta\nu = \frac{c^2}{8\pi\tau\nu^2}\cdot\frac{q_2}{q_1} \quad . \quad . \quad . \quad . \quad (6)$$

Repeating the analysis of Paper I., we find that the equation expressing that the transitions $1\to 2$ balance the transitions $2\to 1$ (equation of radiative equilibrium) is

$$n_1\int_0^\pi I_\nu(\theta)\sin\theta d\theta = n_2(q_1/q_2)\int_0^\pi (I_\nu(\theta)+\sigma)\sin\theta d\theta \quad . \quad (7)$$

where

$$\sigma = 2h\nu^3/c^2,$$

$I_\nu(\theta)$ is the intensity of radiation in direction $\theta$ and $n_1$, $n_2$ are the numbers of atoms per cm.$^3$ in the states 1 and 2 respectively.

The equation determining the rate of change of intensity of radiation in a direction making $\theta$ with the $x$-axis is

$$\frac{\cos\theta}{\alpha}\frac{dI_\nu(\theta)}{dx} = -n_1 I_\nu(\theta) + n_2(q_1/q_2)(I_\nu(\theta)+\sigma) \quad . \quad . \quad (8)$$

In these equations the term in $n_2 I_\nu(q_2/q_1)$ represents the stimulated radiation. As before, we have the integral of the constancy of net flux, in the form

$$2\pi\int_0^\pi I_\nu(\theta)\cos\theta\sin\theta d\theta = \pi F_\nu, \quad . \quad . \quad . \quad (9)$$

$\pi F_\nu\Delta\nu$ being the net flux and $F_\nu$ being the residual intensity in the line.

We require also the equation of mechanical equilibrium. The force due to radiation pressure, taking account of the recoil momentum resulting from the stimulated emissions as well as the momentum from the absorptions, is found to be

$$\frac{\pi\alpha F_\nu\Delta\nu}{c}[n_1 - n_2(q_1/q_2)],$$

and hence if $p$ is the pressure, $m$ the mass of the atom, and $g$ the value of gravity,

$$\frac{dp}{dx} = -(n_1+n_2)mg + \frac{\pi\alpha F_\nu\Delta\nu}{c}[n_1 - n_2(q_2/q_1)] \quad . \quad . \quad (10)$$

3. *Solution of the Radiation Equations.*—We employ the Schuster-Schwarzschild approximation. Replacing $I_\nu(\theta)$ by its mean values in

$(0, \frac{1}{2}\pi)$ and in $(\frac{1}{2}\pi, \pi)$ respectively, and denoting the mean values by $I_\nu$ and $I'_\nu$, equation (7) becomes

$$n(I_\nu + I'_\nu) = 2\sigma n_2(q_1/q_2) \quad . \quad . \quad . \quad (11)$$

where we have written for brevity

$$n = n_1 - n_2(q_1/q_2).$$

Also multiplying (8) by $\sin\theta d\theta$ and integrating in turn from 0 to $\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ to $\pi$, we have approximately

$$\tfrac{1}{2}\frac{dI_\nu}{dx} = \alpha[-nI_\nu + \sigma n_2(q_1/q_2)] \quad . \quad . \quad . \quad (12)$$

$$\tfrac{1}{2}\frac{dI'_\nu}{dx} = \alpha[nI'_\nu - \sigma n_2(q_1/q_2)] \quad . \quad . \quad . \quad (13)$$

Lastly (9) becomes

$$I_\nu - I'_\nu = F_\nu \, . \quad . \quad . \quad . \quad . \quad (14)$$

Eliminating $I_\nu$ and $I'_\nu$ by means of (11) and (14), equations (12) and (13) give

$$\frac{dI_\nu}{dx} = \frac{dI'_\nu}{dx} = -n\alpha F_\nu \quad . \quad . \quad . \quad . \quad (15)$$

The solution can now be carried out as in Paper I., but it is possible to make the analysis a little more direct and at the same time establish a slightly more general result as follows.* We introduce the optical thickness $t$, defined by

$$t = \int_x^\infty n\alpha dx, \quad dt = -n\alpha dx \quad . \quad . \quad . \quad (16)$$

Equations (15) now have for their solution

$$I_\nu = F_\nu(1+t), \quad I' = F_\nu t \quad . \quad . \quad (17), (18)$$

whence from (11)

$$\frac{n_2}{n} = \frac{q_2}{q_1} \cdot \frac{F_\nu}{\sigma}(\tfrac{1}{2}+t) \quad . \quad . \quad . \quad . \quad (19)$$

This important formula determines the allocation of the atoms amongst the states 1 and 2 as a function of the optical thickness $t$. We see that $t$ not only measures the optical depth *down to* the level $x$; it also measures the ratio $n_2/n$ *at* the level $x$.

4. *Residual Intensity.*—We now write

$$n_1 + n_2 = n + n_2[1 + (q_1/q_2)] \quad . \quad . \quad . \quad (20)$$

Then inserting in (10) and using (6),

$$\frac{dp}{dx} = -nmg\left[1 + \frac{F_\nu}{\sigma}(\tfrac{1}{2}+t)\left(1+\frac{q_2}{q_1}\right) - \frac{cF_\nu}{8\nu^2\tau mg} \cdot \frac{q_2}{q_1}\right] . \quad . \quad (21)$$

* The parameter $\eta$ used in the previous paper is equal to $2t$. This follows from the equation $\frac{1}{2}d\eta/dx = -n\alpha$ which it was shown to satisfy, combined with $\eta \to 0$ as $x \to \infty$.

If the atmosphere is "fully supported" by radiation pressure, the coefficient of $nmg$ must tend to zero at great heights, *i.e.* as $t \to 0$. Hence

$$0 = 1 + \tfrac{1}{2}\frac{F_\nu}{\sigma}\left(1 + \frac{q_2}{q_1}\right) - \frac{cF_\nu}{8\nu^2\tau mg} \cdot \frac{q_2}{q_1}, \quad . \quad . \quad . \quad (22)$$

whence

$$\frac{F_\nu}{\sigma} = \frac{q_1/q_2}{s - \tfrac{1}{2}(1 + q_1/q_2)} \quad . \quad . \quad . \quad . \quad (23)$$

on putting

$$s = \frac{c\sigma}{8\nu^2\tau mg} = \frac{h\nu}{4\tau cmg} \quad . \quad . \quad . \quad . \quad (24)$$

5. *Pressure.*—To determine the pressure $p$ as a function of $t$, we simplify (21) by means of (22) and insert $dt = -n\alpha dx$. We then find

$$\frac{dp}{dt} = \frac{mg}{\alpha}\left(1 + \frac{q_2}{q_1}\right)\frac{F_\nu}{\sigma}t,$$

whence

$$p = \tfrac{1}{2}\left(1 + \frac{q_2}{q_1}\right)\frac{mg}{\alpha} \cdot \frac{F_\nu}{\sigma}t^2 \quad . \quad . \quad . \quad . \quad (25)$$

This relation is independent of the temperature distribution in the chromosphere. The proportionality $p \propto t^2$ is characteristic of an atmosphere fully supported by radiation pressure, as opposed to $p \propto t$ for an atmosphere in gravitational equilibrium, or only partially supported.*

The total mass down to a given optical depth can also be found without knowing the temperature distribution. From (20), (19), and (16) we find

$$\int_x^\infty (n_1 + n_2)dx = \frac{t}{\alpha}\left[1 + \tfrac{1}{2}\frac{F_\nu}{\sigma}\left(1 + \frac{q_2}{q_1}\right)(1 + t)\right],$$

which reduces to

$$\int_x^\infty (n_1 + n_2)dx = \frac{t[s + \tfrac{1}{2}(1 + q_1/q_2)t]}{\alpha[s - \tfrac{1}{2}(1 + q_1/q_2)]} \quad . \quad . \quad . \quad (26)$$

6. *Density Law.*—To find $x$ as a function of $t$ it is necessary for the first time to introduce the temperature, for we have to connect $p$ with the number of atoms per cm.$^3$ For simplicity we take the temperature constant and equal to $T_0$, as far as concerns the molecular velocities. There are $2(n_1 + n_2)$ particles per unit volume, counting the electrons as well as the $Ca^+$ ions, and so

$$p = 2(n_1 + n_2)kT_0 \quad . \quad . \quad . \quad . \quad (27)$$

Hence from (25), the density is given by

$$\frac{\rho}{m} = n_1 + n_2 = \frac{mg}{4\alpha kT_0}\left(1 + \frac{q_2}{q_1}\right)\frac{F_\nu}{\sigma}t^2 \quad . \quad . \quad . \quad (28)$$

* In equilibrium of the partially supported type, if $k$ is the mass absorption coefficient, and $g'$ the value of effective gravity, we have the standard equation $p = (g'/k)t$. When the atmosphere is fully supported, this breaks down because $g' \to 0$ at great heights, and we get (25) instead.

Knowing $(n_1+n_2)$ as a function of $t$, we can find $n$ as a function of $t$ from (20) and (19). Inserting this in $dt=-nadx$ and integrating, we find

$$\frac{mg}{4kT_0}(x+x_0)=-\log t+\frac{s}{1+q_1/q_2}\cdot\frac{1}{t} \quad . \quad . \quad (29)$$

where $x_0$ is a constant of integration.

Equations (23), (25), (26), (28), and (29) are the required modifications of the corresponding formulæ in Paper I. It is convenient to have approximations to them. It is easily seen that $F_\nu/\sigma$ is small compared with unity. For if the intensity-ratio in the line is $r$, and the continuous spectrum is black radiation of temperature $T_1$, then $F_\nu/\sigma = r(e^{h\nu/kT_1}-1)^{-1}$, which is always small compared with unity in practice. It follows from (23) that $s$ is large compared with unity, as can also be verified directly. We have then the approximations

$$F_\nu=\frac{q_1}{q_2}\cdot\frac{8\nu^2\tau mg}{c} \quad . \quad . \quad . \quad . \quad (30)$$

$$\frac{\rho}{m}=n_1+n_2=\frac{mg}{4\alpha kT_0}\left(1+\frac{q_1}{q_2}\right)\frac{t^2}{s} \quad . \quad . \quad . \quad (31)$$

$$x+x_0=\frac{4kT_0}{mg}\cdot\frac{s}{1+q_1/q_2}\cdot\frac{1}{t} \quad . \quad . \quad (32)$$

Eliminating $t$ between the last two, we have the density law in the form

$$\frac{\rho}{m}=n_1+n_2=\frac{4kT_0}{\alpha mg}\cdot\frac{s}{1+q_1/q_2}\cdot\frac{1}{(x+x_0)^2} \quad . \quad . \quad (33)$$

The latter can be simplified by inserting

$$\frac{s}{\alpha}=\frac{\pi\sigma\Delta\nu}{cmg}\cdot\frac{q_1}{q_2}.$$

We see that the main character of the equilibrium is unaltered. The density varies, as before, as the square of the height above the level $x=-x_0$. The numerical consequences of (30) have been already discussed in § 1 (1).

## II. *Effect of the Doublet Character of the $1\pi$ Level.*

7. *Formation of the Equations.*—We proceed to take into account the circumstance that the $1\pi$ level consists actually of two levels, the $1\pi_1$ and $1\pi_2$ levels. Transitions from the $1\sigma$ level may occur to either of these states, with the absorption of H or of K radiation respectively. We continue to assume, however, that transitions from one of the $1\pi$ states can only occur to the $1\sigma$ state.

We write $\nu'$, $\nu''$ for the frequencies; $q_2'$, $q_2''$ for the weights; $F_{\nu'}$, $F_{\nu''}$ for the residual intensities; $n_2'$, $n_2''$ the numbers per unit volume in the states $2'$ and $2''$; $\alpha'$, $\alpha''$ for the absorption coefficients; $\tau'$, $\tau''$ the "average lives"; $\Delta\nu'$, $\Delta\nu''$ the widths; $t'$, $t''$ the optical depths for the radiations $\nu'$ and $\nu''$; and we define $\sigma'$, $\sigma''$ and $n'$, $n''$ similarly to the $\sigma$ and $n$ of Section I.

The chief difference is that whilst we have now two sets of radiation equations, we have still only one equation of mechanical equilibrium, which now, however, contains additional terms for the atoms in the $2''$ state and the radiation pressure arising from $F_{\nu''}$. Thus we find from the separate radiative equilibria in $\nu'$ and $\nu''$

$$\frac{n_2'}{n'}=\frac{q_2'}{q_1}\cdot\frac{F_{\nu'}}{\sigma'}(\tfrac{1}{2}+t'),\quad \frac{n_2''}{n''}=\frac{q_2''}{q_1}\cdot\frac{F_{\nu''}}{\sigma''}(\tfrac{1}{2}+t'') \quad . \quad (34),\ (35)$$

$$dt'=-n'\alpha' dx,\quad dt''=-n''\alpha'' dx' \quad . \quad . \quad (36),\ (37)$$

whilst for the mechanical equilibrium

$$\frac{dp}{dx}=-(n_1+n_2'+n_2'')mg+n'\frac{\pi\alpha' F_{\nu'}\Delta\nu'}{c}+n''\frac{\pi\alpha'' F_{\nu''}\Delta\nu''}{c} \quad . \quad (38)$$

8. *Residual Intensities.*—We have

$$n_1=n'+n_2'(q_1/q_2')=n'[1+(F_{\nu'}/\sigma')(\tfrac{1}{2}+t')] \quad . \quad . \quad (39)$$

and similarly

$$n_1=n''[1+(F_{\nu''}/\sigma'')(\tfrac{1}{2}+t'')] \quad . \quad . \quad . \quad (40)$$

With the aid of (34), (35), (39), and (40) we can write (38) in the form

$$\frac{dp}{dx}=-n_1 mg\left[1+\frac{(q_2'/q_1)(F_{\nu'}/\sigma')(\tfrac{1}{2}+t'-s')}{1+(F_{\nu'}/\sigma')(\tfrac{1}{2}+t')}+\frac{(q_2''/q_1)(F_{\nu''}/\sigma'')(\tfrac{1}{2}+t''-s'')}{1+(F_{\nu''}/\sigma'')(\tfrac{1}{2}+t'')}\right] \quad (41)$$

where

$$s'=\frac{h\nu'}{4\tau' cmg},\quad s''=\frac{h\nu''}{4\tau'' cmg}.$$

For an atmosphere fully supported by radiation pressure, the coefficient of $n_1 mg$ must tend to zero as $t'$, $t''\to 0$. Hence

$$0=1+\frac{(q_2'/q_1)(F_{\nu'}/\sigma')(\tfrac{1}{2}-s')}{1+\tfrac{1}{2}(F_{\nu'}/\sigma')}+\frac{(q_2''/q_1)(F_{\nu''}/\sigma'')(\tfrac{1}{2}-s'')}{1+\tfrac{1}{2}(F_{\nu''}/\sigma'')} \quad . \quad (42)$$

From the experimental evidence for the summation rules, as shown by Ornstein and Burger,* we have $A_{2'\to 1}=A_{2''\to 1}$, and hence $\tau'=\tau''=\tau$, say. In (42) we can approximate by writing $\nu$ for $\nu'$ and $\nu''$ in $\sigma'$, $\sigma''$ and in $s'$, $s''$.

Further neglecting $\frac{1}{2}$ in comparison with $s$, and $F_{\nu'}/\sigma'$, $F_{\nu''}/\sigma''$ in comparison with unity, (42) gives approximately

$$q_2' F_{\nu'}+q_2'' F_{\nu''}=\frac{8\nu^2\tau mg}{c}q_1 \quad . \quad . \quad . \quad (43)$$

As pointed out in the introduction, the condition of mechanical equilibrium provides only one relation between the residual intensities. Between them they share the task of supporting the atom against gravity. The relative extents to which they do this must depend on the relative supply of $\nu'$ and $\nu''$ radiation from below. We are no longer able to calculate the residual intensities in the separate components from consideration of the high-level atoms only, but must obtain

* *Loc. cit.*

them by tracing back the pencils of radiation to the base of the atmosphere.

9. *Solution taking into Account the Conditions to be Satisfied at the Lower Boundary.*—From (36) and (37) we have $dt'/dt''=n'\alpha'/n''\alpha''$, and the ratio $n'/n''$ follows from (39) and (40). We find then

$$[1+(F_{\nu'}/\sigma')(\tfrac{1}{2}+t')]dt'/\alpha'=[1+(F_{\nu''}/\sigma'')(\tfrac{1}{2}+t'')]dt''/\alpha'',$$

which integrates in the form

$$\frac{t'[2+(F_{\nu'}/\sigma')(1+t')]}{\alpha'}=\frac{t''[2+(F_{\nu''}/\sigma'')(1+t'')]}{\alpha''} \quad . \quad . \quad (44)$$

Now the outward intensities $I_{\nu'}$, $I_{\nu''}$ at the heights $t'$, $t''$ respectively are given by

$$I_{\nu'}=F_{\nu'}(1+t'), \quad I_{\nu''}=F_{\nu''}(1+t''),$$

whence eliminating $t'$ and $t''$

$$\frac{(I_{\nu'}-F_{\nu'})(I_{\nu'}+2\sigma')}{\alpha'\sigma'F_{\nu'}}=\frac{(I_{\nu''}-F_{\nu''})(I_{\nu''}+2\sigma'')}{\alpha''\sigma''F_{\nu''}} \quad . \quad . \quad (45)$$

This relation connects the intensities $I_{\nu'}$, $I_{\nu''}$ at any particular depth. We apply it to the photospheric depth. Here we may assume $I_{\nu'}$, $I_{\nu''}$ to be each the black-body intensity for the photospheric effective temperature $T_1$, so that

$$I_{\nu'}=\sigma'(e^{h\nu'/kT_1}-1)^{-1}, \quad I_{\nu''}=\sigma''(e^{h\nu''/kT_1}-1)^{-1}.$$

Inserting these in (45) we get a second relation between the residual intensities $F_{\nu'}$ and $F_{\nu''}$.

Write $r'=F_{\nu'}/I_{\nu'}$, $r''=F_{\nu''}/I_{\nu''}$. Then (45) becomes

$$\left(\frac{1}{r'}-1\right)\frac{2+I_{\nu'}/\sigma'}{\alpha'}=\left(\frac{1}{r''}-1\right)\frac{2+I_{\nu''}/\sigma''}{\alpha''} \quad . \quad . \quad . \quad (46)$$

We may neglect $I_{\nu'}/\sigma'$, $I_{\nu''}/\sigma''$ in comparison with unity. Now we have, applying (6),

$$\alpha'\Delta\nu'=\frac{c^2}{8\pi\tau'\nu'^2}\cdot\frac{q_2'}{q_1}, \quad \alpha''\Delta\nu''=\frac{c^2}{8\pi\tau''\nu''^2}\cdot\frac{q_2''}{q_1}.$$

The widths $\Delta\nu'$, $\Delta\nu''$ on the present theory arise from the dispersion of the narrow proper frequency-range of the atom due to outside influences. It is reasonable to assume that these will be the same for the $1\pi_1$ and $1\pi_2$ states, and hence we may take $\Delta\nu'=\Delta\nu''$. Also $\tau'=\tau''$. Hence approximately

$$\frac{\alpha'}{\alpha''}=\frac{q_2'}{q_2''}.$$

Neglecting unity in comparison with $1/r'$, $1/r''$ in (46) we then have

$$q_2'F_{\nu'}=q_2''F_{\nu''} \quad . \quad . \quad . \quad . \quad (47)$$

Subject to the various approximations which have been mentioned, we have thus proved that for an atmosphere the residual intensities obey the same rule as would be inferred from a straightforward applica-

tion of the summation rules of Burger and Dorgelo. Equations (47) and (43) suffice to determine $F_{\nu'}$ and $F_{\nu''}$. The question of the experimental confirmation of (47) has been dealt with in the introduction.

10. *Pressure.*—Equation (41) may be integrated by reducing the right-hand side to a perfect differential form in $dt'$ and $dt''$. The result is

$$p=mg\left[\frac{1+s'(F_{\nu'}/\sigma')}{1+\frac{1}{2}(F_{\nu'}/\sigma')}\frac{F_{\nu'}}{\sigma'}\frac{q_2'}{q_1}\frac{\frac{1}{2}t'^2}{\alpha'}+\frac{1+s''(F_{\nu''}/\sigma'')}{1+\frac{1}{2}(F_{\nu''}/\sigma'')}\frac{F_{\nu''}}{\sigma''}\frac{q_2''}{q_1}\frac{\frac{1}{2}t''^2}{\alpha''}\right],$$

where, it may be recalled, $t'$ and $t''$ are connected by the relation (44). The formula for $p$ reduces approximately to

$$p=\tfrac{1}{2}mg\left[\left(\tfrac{1}{2}+\frac{q_2'}{q_1}\right)\frac{F_{\nu'}}{\sigma'}\frac{t'^2}{\alpha_1}+\left(\tfrac{1}{2}+\frac{q_2''}{q_1}\right)\frac{F_{\nu''}}{\sigma''}\frac{t''^2}{\alpha''}\right],$$

which is the required generalisation of (25). Equation (44) shows that approximately $t'/\alpha'=t''/\alpha''$. Hence we have $p\propto t'^2$. The law of density $\rho\propto(x+x_0)^{-2}$ is unaltered.

## III. *Effect of the Metastable* 1δ *State.*

11. *Formation of the Equations.*—We denote the 1σ, 1π, and 1δ states by the symbols 1, 2, 1′ respectively. Both the 1σ and 1δ state correspond to energies exceeding the 1π state, but the only permissible transitions are $1\gtrless 2$ and $1'\gtrless 2$. We let $\nu$, $\nu'$ denote the corresponding frequencies; $F_\nu$, $F_{\nu'}$ the residual intensities; $\alpha$, $\alpha'$ the atomic absorption coefficients for atoms in the 1 and 1′ states; $\Delta\nu$, $\Delta\nu'$ the widths of the lines; $q_1$, $q_1'$, $q_2$ the weights; $n_1$, $n_1'$, $n_2$ the numbers of atoms per cm.$^3$ in the three states. We define $\tau$, $\tau'$ as the reciprocals of the Einstein probability coefficients $A_{2\to1}$, $A_{2\to1'}$; the average life in the excited state is now $(A_{2\to1}+A_{2\to1'})=\tau\tau'/(\tau+\tau')$, but it is convenient to retain the term "average life" for the quantities $\tau$, $\tau'$ separately. For brevity we write $\sigma$ for $2h\nu^3/c^2$, $\sigma'$ for $2h\nu'^3/c^2$, and we put

$$n=n_1-n_2(q_2/q_1),\quad n'=n_1'-n_2(q_2/q_1') \qquad . \qquad (48),\ (49)$$

It is easily found that the usual radiation equations hold for each frequency separately. Putting

$$dt=-n\alpha dx,\quad dt'=-n'\alpha' dx \qquad . \qquad . \qquad (50),\ (51)$$

we find then, as in Section I.,

$$\frac{n_2}{n}=\frac{q_2}{q_1}\cdot\frac{F_\nu}{\sigma}(\tfrac{1}{2}+t),\quad \frac{n_2}{n'}=\frac{q_2}{q_1'}\cdot\frac{F_{\nu'}}{\sigma'}(\tfrac{1}{2}+t') \qquad . \qquad (52),\ (53)$$

The equation of mechanical equilibrium is

$$\frac{dp}{dx}=-(n_1+n_1'+n_2)mg+\frac{\pi\alpha F_\nu\Delta\nu}{c}n+\frac{\pi\alpha' F_{\nu'}\Delta\nu'}{c}n' \qquad . \qquad (54)$$

By the theory of radiation, we have

$$\pi\alpha\Delta\nu=\frac{c^2}{8\nu^2\tau}\cdot\frac{q_2}{q_1},\qquad \pi\alpha'\Delta\nu'=\frac{c^2}{8\nu'^2\tau'}\cdot\frac{q_2}{q_1'} \qquad . \qquad (55),\ (56)$$

12. *Residual Intensities.*—We have

$$n_1+n_1'+n_2=n+n'+n_2\left(1+\frac{q_1+q_1'}{q_2}\right) \quad . \quad . \quad (57)$$

Hence (54) may be written, on making use of (52) and (53),

$$\frac{dp}{dx}=-n_2mg\left[1+\frac{q_1+q_1'}{q_2}+\frac{q_1/q_2}{(F_\nu/\sigma)(\frac{1}{2}+t)}+\frac{q_1'/q_2}{(F_{\nu'}/\sigma')(\frac{1}{2}+t')}\right.$$
$$\left.-\frac{c\sigma}{8\nu^2\tau mg(\frac{1}{2}+t)}-\frac{c\sigma'}{8\nu'^2\tau' mg(\frac{1}{2}+t')}\right] \quad (58)$$

If the atmosphere is "fully supported" by radiation pressure the coefficient of $n_2mg$ must tend to zero as $t\to 0$. Hence

$$c=1+\frac{q_1+q_1'}{q_2}+2\sigma\left(\frac{q_1/q_2}{F_\nu}-\frac{c}{8\nu^2\tau mg}\right)+2\sigma'\left(\frac{q_1'/q_2}{F_{\nu'}}-\frac{c}{8\nu'^2\tau' mg}\right) \quad (59)$$

This is the full form, of which an approximation was given in the introduction, equation (2) above. As there stated, the condition of mechanical equilibrium of the high-level atoms provides one relation only. To obtain a second one between two unknowns, $F_\nu$ and $F_{\nu'}$, we have to consider the external radiation incident on the layer of gas from below.

It happens, however, that for the $Ca^+$ lines we are considering, the term in (59) involving the unknown $F_{\nu'}$ can be neglected. For if $r$, $r'$ are the intensity-ratios in the lines, and the assumption is made that the radiation incident from below, which constitutes the continuous background, is black radiation of temperature $T_1$, then by Planck's formula

$$F_\nu=\frac{r\sigma}{e^{h\nu/kT_1}-1}, \qquad F_{\nu'}=\frac{r'\sigma'}{e^{h\nu'/kT_1}-1},$$

and hence

$$\frac{\sigma'/F_{\nu'}}{\sigma/F_\nu}=\frac{r}{r'}\cdot\frac{e^{h\nu'/kT}-1}{e^{h\nu/kT}-1}.$$

For $\lambda=3950$, $\lambda'=8580$, $T_1=6000°$, the right-hand side is equal to $0{\cdot}035/r/r'$. Assuming that the line due to the metastable state is not more intense than that due to the normal state, we have $r/r'\leqslant 1$ (our estimate made below gives in fact $r/r'=0{\cdot}11/0{\cdot}4=0{\cdot}3$). With an error of 1 or 2 per cent., then, we may neglect the term in $F_{\nu'}$ in (59). It then gives approximately

$$F_\nu=\frac{q_1}{q_2}\cdot\frac{8\nu^2\tau mg}{c}\cdot\frac{1}{1+(\nu'\tau/\nu\tau')} \quad . \quad . \quad . \quad (60)$$

We should expect transitions from the $1\pi$ to the $1\delta$ state to be less probable than transitions from the $1\pi$ to the $1\sigma$ state, *i.e.* $A_{2\to 1}>A_{2\to 1'}$, or $\tau'>\tau$. The exact value of the ratio $A_{2\to 1}/A_{2\to 1'}=\tau'/\tau$ could be determined by precise measures of the relative intensities of the principal doublet and the diffuse doublet in the laboratory. Until such measures are available, we cannot evaluate $\tau$ with any accuracy from (60). But to see the order of magnitude of the correction involved, we may put

$\tau'=\tau$. We then have $\nu'\tau/\nu\tau'=3950/8580=0{\cdot}46$. Hence for given $F_\nu$, the value of $\tau$ deduced from the simple theory of Section I. requires multiplying by the factor 1·46, making it $\tau=1{\cdot}46\times1{\cdot}8\times10^{-8}=2{\cdot}6\times10^{-8}$ sec. The true mean life $\tau\tau'/(\tau+\tau')$, assuming again $\tau'=\tau$, is then $1{\cdot}3\times10^{-8}$ sec. Actually we may expect the factor to be less than 1·46.

13. *Determination of the Residual Intensities, taking Account of the Conditions at the Lower Boundary of the Atmosphere.*—We first connect the optical depths $t$ and $t'$ with one another. We then retrace the intensity of outward radiation down to the photosphere in the two frequencies $\nu$ and $\nu'$, and the condition is that at this depth these intensities must equal the given photospheric intensities.

From (52) and (53) we find $n/n'$, and we insert this in the relation $dt/dt'=n\alpha/n'\alpha'$. The result is

$$\frac{(F_\nu/\sigma)(\frac{1}{2}+t)dt}{q_1\alpha}=\frac{(F_{\nu'}/\sigma')(\frac{1}{2}+t')dt'}{q_1'\alpha'} \quad . \quad . \quad . \quad (61)$$

This integrates, giving

$$\frac{(F_\nu/\sigma)t(1+t)}{q_1\alpha}=\frac{(F_{\nu'}/\sigma')t'(1+t')}{q_1'\alpha'} \quad . \quad . \quad . \quad (62)$$

But the outward intensities $I_\nu$ and $I_{\nu'}$ at any level are given by

$$I_\nu=F_\nu(1+t), \quad I_{\nu'}=F_{\nu'}(1+t').$$

Hence

$$\frac{(I_\nu/\sigma)(I_\nu-F_\nu)}{\alpha q_1 F_\nu}=\frac{(I_{\nu'}/\sigma')(I_{\nu'}-F_{\nu'})}{\alpha' q_1' F_{\nu'}} \quad . \quad . \quad (63)$$

This is a relation connecting the outward intensities $I_\nu$ and $I_{\nu'}$ at any level. In particular it will hold at the base of the atmosphere, supposing the atmosphere to be bounded below by a surface radiating with given intensity. Letting now $I_\nu$, $I_{\nu'}$ refer to the photospheric level and introducing the intensity-ratios $r$ and $r'$ defined above, we have from (63)

$$\frac{r^{-1}-1}{r'^{-1}-1}=\frac{\alpha q_1}{\alpha' q_1'}\cdot\frac{I_{\nu'}/\sigma'}{I/\sigma} \quad . \quad . \quad . \quad . \quad (64)$$

Equations (63) and (59) may be regarded as a pair of equations for the unknowns $F_\nu$ and $F_{\nu'}$. We have shown that the term in (59) in $F_{\nu'}$ may be neglected for the $Ca^+$ spectrum, and we can therefore substitute the value of $F_\nu$ given by (60) in (63), thus determining $F_{\nu'}$. Or alternatively, taking $r$ for the H and K doublet as given by observation, (64) allows us to predict the intensity-ratio $r'$ in the infra-red diffuse doublet.

But it is clear that equations (63) or (64) are on a very different footing from (59). Relation (59), the condition of mechanical equilibrium of the high-level atoms, involves besides the residual intensities only atomic constants relating to the atoms in question together with the value of gravity. It is independent of the structure of the atmosphere formed by the atoms below. Relation (65) is not so independent. In substituting photospheric values for $I_\nu$ and $I_{\nu'}$ we are assuming that

the various conditions postulated hold down to the photospheric level. This can scarcely be true, for admixture with other atoms, collisions, and change of ionization will interfere.

Nevertheless it is worth while making a tentative numerical calculation on the basis of (64). Formula (64) is applicable rigorously to the case of an atmosphere of $Ca^+$ atoms above a plane radiating surface radiating with intensity $I_\nu$ in $\nu$ and $I_{\nu'}$ in $\nu'$. This model, though not an accurate description of the way the chromosphere and reversing layer fade into the photosphere, is worth studying for the indications it affords. Relations (63) and (64) are purely radiation equations; they involve no assumptions as to the mechanical equilibrium of the atmosphere.

To make use of (64) we require $\alpha/\alpha'$, the ratio of the atomic absorption coefficients. We have from (55) and (56)

$$\frac{q_1 a \Delta\nu}{q_1' a' \Delta\nu'} = \frac{\nu'^2 \tau'}{\nu^2 \tau}. \quad . \quad . \quad . \quad . \quad . \quad (65)$$

The atoms being exposed to much the same broadening influences, we may put $\Delta\nu = \Delta\nu'$. Hence from (64)

$$\frac{r^{-1}-1}{r'^{-1}-1} = \frac{\tau'\nu'^2}{\tau\nu^2} \cdot \frac{e^{h\nu/kT}-1}{e^{h\nu'/kT}-1} \quad . \quad . \quad . \quad (66)$$

For $\lambda = 3950$, $\lambda' = 8580$, $T_1 = 6000°$, the coefficient of $\tau'/\tau$ in (66) comes to 5·93. For the H and K lines (Schwarzschild's measure) $r = 0{\cdot}11$. Solving (66) we find

$$r' = \frac{5{\cdot}93}{5{\cdot}93 + 8{\cdot}1(\tau/\tau')}.$$

Taking $\tau' = \tau$ as before, for definiteness, we get $r' = 0{\cdot}42$. If $\tau' > \tau$, a bigger value of $r'$ would result.

Thus the diffuse $Ca^+$ lines in the infra-red should be weaker in the Fraunhofer spectrum than the H and K lines, in the sense that they should have about 4 times the ratio *residual intensity/adjacent background*. No measures appear to exist with which to check this estimate.

14. *Brightness in the Flash Spectrum.*—The same calculation allows us to estimate the relative brightness of the H, K doublet and the X-group in the flash spectrum. A line faint in the Fraunhofer spectrum (hence possessing a relatively large residual intensity) should be more intense in the flash spectrum. This was found to be the case by Mitchell in his 1905 spectrum—the low-lying vapours corresponding to faint Fraunhofer lines gave short bright arcs of considerable intensity in the flash spectrum. The ratio of the residual intensities is

$$\frac{F_{\nu'}}{F_\nu} = \frac{r' I_{\nu'}}{r I_\nu}.$$

Now $I_\nu/I_{\nu'} = 2{\cdot}7$, $r'/r = 0{\cdot}4/0{\cdot}11 = 3{\cdot}8$. Hence $F_{\nu'}/F_\nu = 10$. This should be the ratio of the bright-line intensities in the flash, since the tangentially emergent light at any level is proportional to the normally emergent, provided the line of sight intercepts a sufficient optical thick-

ness to give complete opacity. At great heights, where this condition is not satisfied, the ratio should be $\nu' A_{2\to 1'}/\nu A_{2\to 1} = \nu'\tau/\nu\tau'$, as explained in § 1. For $\tau' = \tau$, this is 0·46. If $\tau'$ is comparable with $\tau$, the intensities at great heights should be comparable, and the arcs should be of comparable length in the flash. If, however, $\tau'$ is much longer than $\tau$, the ratio will be considerably less than unity, and the arcs for the diffuse lines will be shorter than for H and K. At lower levels, as we have seen, the infra-red group should give more intense lines than H and K.

15. *Pressure and Density.*—The modifications of the pressure and density formulæ necessary to take account of the existence of the metastable 1δ level are obtained as follows:—

Using (59) to simplify (58), we have

$$\frac{dp}{dx} = -n_2 mg\left[\frac{2t(q_1/q_2)}{(F_\nu/\sigma)(\frac{1}{2}+t)}\left(\frac{cF_\nu(q_2/q_1)}{8\nu^2\tau mg} - 1\right) + \frac{2t'(q_2/q_1')}{(F_{\nu'}/\sigma')(\frac{1}{2}+t')}\left(\frac{cF_{\nu'}(q_2/q_1')}{8\nu'^2\tau' mg} - 1\right)\right] \quad (67)$$

But

$$\frac{(q_1/q_2)n_2 dx}{(F_\nu/\sigma)(\frac{1}{2}+t)} = n dx = -\frac{dt}{a},$$

with a similar formula in $t'$. Inserting these in (67) it becomes the sum of two perfect differentials, and gives on integrating

$$p = mg\left[\left(\frac{cF_\nu(q_2/q_1)}{8\nu^2\tau mg} - 1\right)\frac{t^2}{a} + \left(\frac{cF_{\nu'}(q_2/q_1')}{8\nu'^2\tau' mg} - 1\right)\frac{t'^2}{a'}\right] \quad . \quad (68)$$

This is the required modification of (25), to which it reduces if the second term is omitted, in virtue of (59).

To find the density we assume a uniform temperature $T_0$ and put $p = 2(n_1 + n_1' + n_2)kT_0$. It is convenient to write

$$C = \frac{cF_\nu(q_2/q_1)}{8\nu^2\tau mg} - 1, \quad C' = \frac{cF_{\nu'}(q_2/q_1')}{8\nu'^2\tau' mg} - 1,$$

$$D = (q_2/q_1)(F_\nu/\sigma), \quad D' = (q_2/q_1')(F_{\nu'}/\sigma').$$

Relation (59) is then written

$$\frac{C}{D} + \frac{C'}{D'} = \frac{1}{2}\left(1 + \frac{q_1 + q_1'}{q_2}\right).$$

From (68)

$$n_1 + n_1' + n_2 = \frac{mg}{2kT_0}\left(\frac{Ct^2}{a} + \frac{C't'^2}{a'}\right),$$

and from (52) and (53)

$$n_1 + n_1' + n_2 = n + n' + n_2\left(1 + \frac{q_1 + q_1'}{q_2}\right)$$

$$= n_2\left[\frac{1}{D(\frac{1}{2}+t)} + \frac{1}{D'(\frac{1}{2}+t')} + \frac{2C}{D} + \frac{2C'}{D'}\right].$$

We thus obtain $n_2$ in terms of $t$ and $t'$, and hence, from (52), $n$. We

insert this in $dx=-dt/n\alpha$, and arrange the result as a differential form in $dt$ and $dt'$ by means of (61). Hence

$$-\frac{mg}{2k\mathrm{T}_0}dx=\frac{\alpha^{-1}[1+\mathrm{C}(1+2t)]dt+\alpha'^{-1}[1+\mathrm{C}'(1+2t')]dt'}{\alpha^{-1}\mathrm{C}t^2+\alpha'^{-1}\mathrm{C}'t'^2}.$$

This integrates in the form

$$\frac{mg(x+x_0)}{2k\mathrm{T}_0}=-\log\left(\frac{\mathrm{C}t^2}{\alpha}+\frac{\mathrm{C}'t'^2}{\alpha'}\right)-\int^{t,t'}\frac{\alpha^{-1}(1+\mathrm{C})dt+\alpha'^{-1}(1+\mathrm{C}')dt'}{\alpha^{-1}\mathrm{C}t^2+\alpha'^{-1}\mathrm{C}'t'^2} \quad (69)$$

the last being a curvilinear integral in $t$ and $t'$ taken along the path

$$\frac{\mathrm{D}t(1+t)}{\alpha}=\frac{\mathrm{D}'t'(1+t')}{\alpha'}.$$

Equation (69) is the required modification of (29).

The mass down to the level $(t, t')$ is readily found to be given by

$$\int_x^\infty(n_1+n_1'+n)dx=\frac{t[1+\mathrm{C}(1+t)]}{\alpha}+\frac{t'[1+\mathrm{C}'(1+t')]}{\alpha'}. \quad (70)$$

This is the modification of (26).

The usual approximate formulæ follow on noting that C and C′ are small compared with unity. At great heights the density law is found to be $\rho\propto(x+x_0)^{-2}$ as before. The main characteristics of the equilibrium are unaltered.

16. *Summary.*—(1) If atoms of mass $m$, capable of two, and only two, stationary states of statistical weights $q_1$ and $q_2$, are supported against gravity $g$ by the radiation pressure arising from the formation of an absorption line of frequency $\nu$, the residual intensity $\mathrm{F}_\nu$ in the line is given by

$$\mathrm{F}_\nu=\frac{q_1}{q_2}\cdot\frac{8\nu^2\tau mg}{c},$$

approximately, where $\tau$ is the mean life in the excited state and $c$ is the velocity of light.

(2) The factor $q_1/q_2$ leads to an alteration in the estimate of $\tau$ for the $1\pi$ state of the $Ca^+$ atom, as derived in a previous paper using the observed value of $\mathrm{F}_\nu$ for the H and K lines in the solar spectrum. We find now $\tau=1{\cdot}8\times10^{-8}$ sec., instead of $0{\cdot}6\times10^{-8}$ sec.

(3) The modifications in the formulæ for pressure and density are worked out. The general characteristics of the equilibrium are unaltered.

(4) Account is taken of the fact that the $1\pi$ state of the $Ca^+$ atom is double, consisting of the $1\pi_1$ and $1\pi_2$ levels. It is shown that the residual intensity calculated as though the level were single is equal to the weighted mean of the residual intensities in the two components of the doublet, the weights being the statistical weights of the two states. The relative residual intensities should be inversely as the weights; this is in accordance with what might have been predicted from the summation rules of Burger and Dorgelo applied to this case.

When the relative residual intensities are inversely as the weights, the weighted mean of the residual intensities is equal to their harmonic mean.

(5) A more general case of chromospheric equilibrium is worked out in which the atom possesses a metastable state of energy intermediate between the energies of the normal and excited states. This is applied to the metastable $1\delta$ states of the $Ca^+$ atom, which give rise to the lines $\lambda\lambda$8498, 8542, 8662 observed in the infra-red Fraunhofer and flash spectra.

(6) Some of the atoms arriving in the $1\pi$ state must necessarily undergo transitions to the $1\delta$ state. Hence the lines of the infra-red group must be being emitted at all heights at which the H, K lines are emitted. Flash spectra taken with plates equally sensitive in the two regions should show arcs of comparable length for the H and K lines and for the infra-red lines.* This accounts for Curtis and Burns' observation of the predominance in the flash spectrum of the $Ca^+$ lines near $\lambda$8600.

(7) The occurrence of the $1\delta$ levels requires a modification in the previous formula for the residual intensity in the H and K lines. It requires division by a factor $(1+\nu'\tau/\nu\tau')$ where $\nu'$ is the mean frequency of the infra-red lines and $\tau'$ is the average life of the $1\pi$ state with reference to transitions to the $1\delta$ state. Taking $\tau=\tau'$, the value of $\tau$ deduced from observation is now found to be $2{\cdot}6\times10^{-8}$ sec., but since there are now two ways of leaving the $1\pi$ state, the actual mean life in the $1\pi$ state is $\tau\tau'/(\tau+\tau')$, which if $\tau'=\tau$ gives $\tau=1{\cdot}3\times10^{-8}$ sec.

(8) The calculated value of the residual intensity-ratio in the infra-red lines, on the assumption $\tau'=\tau$, is 0·4, calculated as though the $1\delta$ level were single. The calculated value of the residual intensity in the infra-red lines is about 10 times that in H and K, and accordingly they should be more intense than H and K at low levels in the flash. At high levels, where the line of sight intercepts only a small optical thickness, the ratio of the bright-line intensities should be $\nu'\tau/\nu\tau'$.

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## *Effective Wave-lengths of Stars from Dec. 80° to the North Pole.*

By Sir Frank Dyson and E. Martin.

This paper is in continuation of two papers by Mr. Davidson and Mr. Martin in *Monthly Notices*, 1921 December and 1924 April. Photographs were taken with the 30-inch reflector, using a grating of equal bar and space. The reflector was stopped down to 20 inches and exposures of $10^m$, $2^m$, and $24^s$ were given on each plate, providing measurable images of stars down to $11^m{\cdot}0$ photographic. The plates used were Seed Gilt Edge and Ilford Special Rapid.

On later photographs exposures of the region round the North Pole were made every night on at least one plate, in order to correct for the variation in atmospheric absorption on different nights. The photo-

* Provided $\tau'$ is comparable with $\tau$. See (8).

graphs were all taken at the altitude of the Pole and are not corrected to the Zenith.

The number of stars in the zone from Dec. 64° to the North Pole between different limits of wave-length are shown in the following table :—

TABLE I.

*Effective Wave-length.*

| Mag | -4150. | 4151–4200. | 4201–4250. | 4251–4300. | 4301–4350. | 4351–4400. | 4401–4450. | 4451–4500. | 4501– | |
|---|---|---|---|---|---|---|---|---|---|---|
| 11·5– | 0 | 0 | 18 | 40 | 24 | 5 | 18 | 13 | 6 | |
| 11·0–11·4 | 1 | 3 | 31 | 117 | 58 | 29 | 51 | 43 | 22 | |
| 10·5–10·9 | 0 | 2 | 71 | 314 | 179 | 63 | 90 | 93 | 47 | |
| 10·0–10·4 | 1 | 2 | 67 | 421 | 215 | 74 | 104 | 130 | 57 | |
| 9·5–9·9 | 0 | 2 | 40 | 345 | 86 | 38 | 62 | 72 | 47 | |
| 9·0–9·4 | 1 | 2 | 41 | 243 | 76 | 27 | 61 | 56 | 33 | |
| 8·0–8·9 | 1 | 1 | 40 | 232 | 39 | 22 | 49 | 37 | 26 | |
| –7·9 | 0 | 4 | 33 | 111 | 12 | 13 | 24 | 11 | 19 | |
| Totals | | 20 | 341 | 1823 | 689 | 271 | 459 | 455 | 257 | 4315 |
| Percentage | | ·5 | 8 | 42 | 16 | 6 | 11 | 11 | 6 | |

Within the limits of magnitude considered there is little evidence of any change with magnitude in the percentage of stars of different effective wave-lengths. The number of stars whose effective wave-length is less than 4200 is 20, or about ½ per cent. of all the stars. About 8 per cent. have effective wave-length less than 4250. A very large number (42 per cent.) have effective wave-length between 4250 and 4300. The small number (6 per cent.) between 4350 and 4400 may be noticed. The very red stars (effective wave-length greater than 4500) amount to 6 per cent. of the total.

Comparison has been made between the effective wave-length and the Spectral Type given in the Draper Catalogue. The results are as follows :—

TABLE II.

| Spectral Type. | Mean Wave-length. | No. of Stars. | Spectral Type. | Mean Wave-length. | No. of Stars. |
|---|---|---|---|---|---|
| Oe5 | 4074 | 1 | F2 | 4273 | 31 |
| Bo | 4167 | 1 | F5 | 4277 | 51 |
| B2 | 4148 | 2 | F8 | 4287 | 58 |
| B3 | 4147 | 4 | Go | 4313 | 59 |
| B5 | 4174 | 4 | G5 | 4402 | 109 |
| B8 | 4220 | 4 | G8 | 4381 | 1 |
| B9 | 4226 | 13 | Ko | 4461 | 162 |
| Ao | 4252 | 89 | K2 | 4506 | 37 |
| A2 | 4265 | 50 | K5 | 4539 | 12 |
| A3 | 4268 | 26 | Ma | 4543 | 10 |
| A5 | 4269 | 33 | Mb | 4491 | 4 |
| Fo | 4270 | 63 | N | 4572 | 1 |

These figures are very similar to those given in *Monthly Notices*, 1921 December. The effective wave-length obtained in this way changes rapidly with spectral type from Bo to Ao and from Go to M. From Ao to Go the change is small, being only 61 A.U. To obtain a more sensitive index a different photographic plate is required.

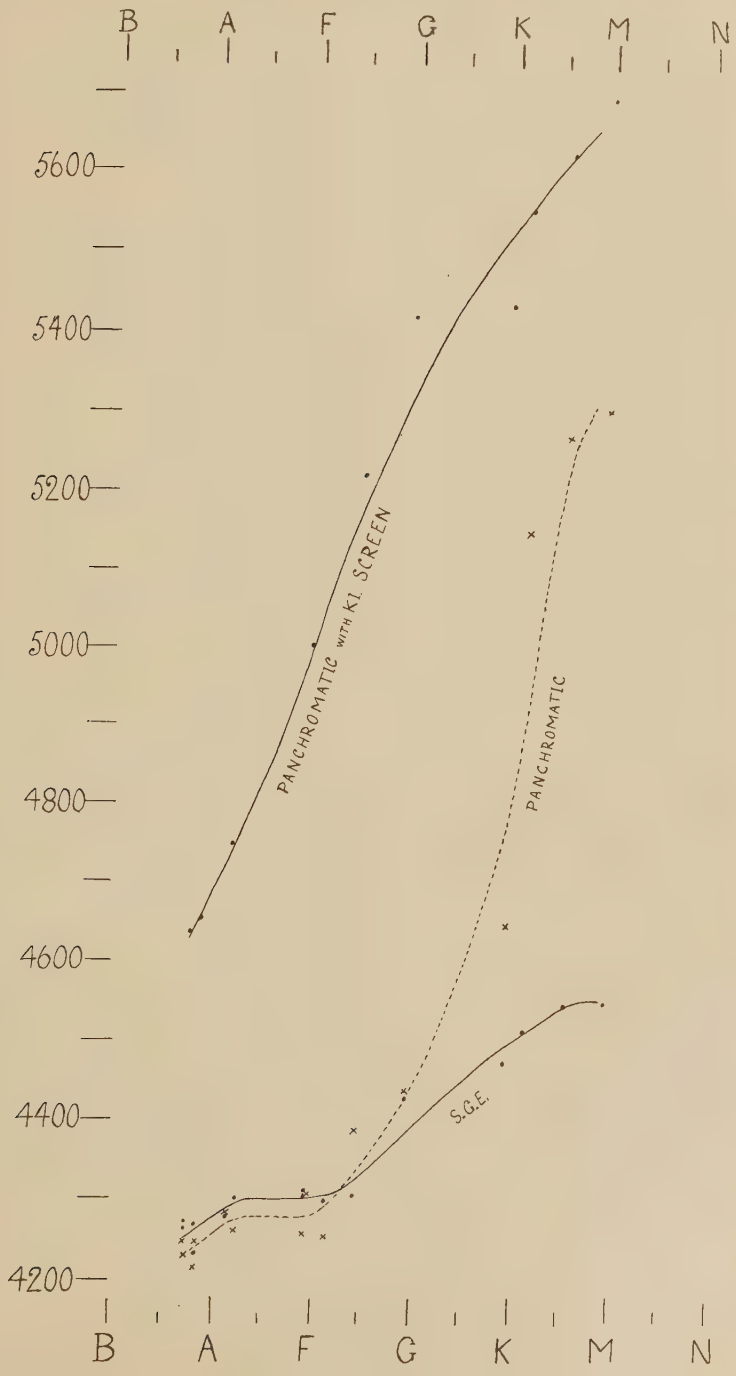
This is shown in the following diagram, where the results of photographing a number of standard stars of different spectral types are exhibited. The plates used were (1) Seed Gilt Edge, (2) Ilford Panchromatic, and (3) Ilford Panchromatic with a K1 screen. It would be possible to effect a still further improvement by the use of a slightly more absorptive screen, say K1½. The panchromatic plate widens the scale for red stars, but is of no advantage for the stars between A and F, and only of slight advantage for Go stars. When, however, the blue light is screened out a very sensitive and approximately linear relationship is given between effective wave-length and spectral type over the range B8 to M. The images are three times as long as those on ordinary plates, but the time of exposure necessary to reach the same limit of magnitude is considerably increased.

For the stars of $8^{m}\cdot0$ and fainter (all magnitudes in the paper are photographic) in the four groups $< 4250$, 4400–4450, 4450–4500, and $> 4500$, whose proper motions are given in the Astrographic Catalogue, vol. iv., the resolved parts of the proper motion in the direction of the Solar Apex, assumed to be at $18^{h}\ 0^{m} + 30°$, were computed. The mean values of these quantities are given (1) when all stars are included; (2) excluding a number of stars of largest proper motion away from the Solar Apex and an equal number of the largest proper motions in the contrary direction, amounting to 3 per cent. of the total number; (3) similarly omitting 10 per cent.

| Effective Wave-length. | Mean of Resolved Part of P.M. | | | Photographic Magnitude. |
|---|---|---|---|---|
| | (1). | (2). | (3). | |
| | ″ | ″ | ″ | m |
| >4500 | ·0053 | ·0052 | ·0051 | 9·88 |
| 4450–4500 | ·0117 | ·0095 | ·0091 | 10·03 |
| 4400–4450 | ·0141 | ·0108 | ·0087 | 9·91 |
| <4250 | ·0083 | ·0078 | ·0082 | 9·62 |

Taking the means of (2) and (3) as representing the parallactic movements of these groups of stars, the mean parallaxes obtained by dividing by 4·1 sin 60°, and Mean Absolute Magnitudes are as follows :—

| Effective Wave-length. | Mean Parallax. | Mean. Absolute Mag. | |
|---|---|---|---|
| | | Photo. | Vis. |
| | ″ | | |
| >4500 | ·00145 | −0·3 | −1·3 |
| 4450–4500 | ·00262 | +2·0 | +1·1 |
| 4400–4450 | ·00275 | +2·1 | +1·4 |
| <4250 | ·00226 | +1·4 | +1·4 |


B
A
F
G
K
M
N
5600
5400
5200
5000
4800
4600
4400
4200
PANCHROMATIC WITH K1 SCREEN
PANCHROMATIC
S.G.E.
B
A
F
G
K
M
N

The stars with wave-length > 4500, amounting to about 6 per cent. of the total number, are all stars of great luminosity. The groups > 4500 and < 4250 appear to be homogeneous groups of stars, as is indicated by comparison of cols. (1), (2), and (3); while the groups 4400–4450 and 4450–4500 seem to contain giants and dwarfs. The comparatively low magnitude of the blue stars as compared with the red is noticeable.

The stars of the spectral types G0, G5, K0, and K2 have been collected in order of wave-length. The proper motions of the stars given in the Astrographic Catalogue, vol. iv., have been used to give an indication of parallax, where this is not already known, by the use of van Rhijn's tables. Also they have been divided into groups and mean parallaxes inferred from the solar motion.

The G0 group contains a considerable proportion of stars of large proper motion; they are equally distributed among the stars above and below the average effective wave-length of the group.

The G5 group contains 107 stars whose effective wave-lengths range from 4246 to 4493, with a mean at 4402. Of the 40 stars whose effective wave-length is less than 4402, there are 8 stars of large proper motion and known parallax whose mean absolute magnitude is $5^{m}\cdot3$. The remaining 32 have a mean absolute magnitude of $2^{m}\cdot2$. The remaining 67 stars with effective wave-length > 4400 show no star with proper motion > ″·150 and their mean absolute magnitude is $0^{m}\cdot2$, as inferred from the solar motion, though van Rhijn's tables make them considerably fainter.

The K0 group contains 158 stars, with a very large range of effective wave-length from 4259 to 4585. There are 43 stars with effective wave-length < 4430, and this list contains 4 stars with proper motion > ″·150 and 8 with proper motion > ″·100; the group with effective wave-length > 4430, containing 115 stars, has no star with proper motion > ″·150 and only 4 with proper motion > ″·100. The mean difference of absolute magnitude between the red and blue stars is only $0^{m}\cdot7$. One might have expected Giants and Dwarfs to be more effectively separated.

The K2 group contains 36 stars whose effective wave-lengths range from 4405 to 4561. The bluer half contains 2 stars with proper motion > ″·150 and of absolute magnitude $5^{m}\cdot5$. The remainder of the group have mean magnitude $1^{m}\cdot8$, as judged from the solar motion, while the group of effective wave-length > 4500 has a mean absolute magnitude of $-0^{m}\cdot8$.

On the whole it may be said that a comparison of effective wave-lengths with the spectral class given in the Draper Catalogue affords clear evidence that the redder stars are of greater luminosity than the bluer stars of the same spectral class, but the differences do not appear to be sufficiently definite to be used as a criterion of luminosity in particular cases.

*Preliminary Note on the Determination of Effective Stellar Temperatures by the "Prism-crossed-by-grating" Method.* By W. M. H. Greaves and C. R. Davidson. (Plate 1.)

(*Communicated by the Astronomer-Royal.*)

According to the theory of black-body radiation, the ratio of the relative intensities of the light emitted by a black-body source in two regions of the spectrum is a function of the absolute temperature, and if this ratio be measured, Planck's formula provides a means of calculating the temperature. If $I_\lambda$ be the light intensity for wave-length $\lambda$, Planck's formula is in fact

$$I_\lambda = C_1 \lambda^{-5} \Big/ \left( e^{\frac{c_2}{\lambda T}} - 1 \right),$$

where $c_2$ is an absolute constant, $C_1$ is independent of $\lambda$, and T is the absolute temperature.

Thus if for two wave-lengths $\lambda\,\lambda'$, the ratio $I_\lambda/I_{\lambda'}$ be measured, T can be determined. The measurement of such a ratio may be effected by comparison with a black-body source of light of known temperature. For if $J_\lambda$, $J_{\lambda'}$ be the corresponding light-intensities for the comparison source, the ratio $J_\lambda/J_{\lambda'}$ can be calculated, and consequently if we can measure the ratios $I_\lambda/J_\lambda$ and $I_{\lambda'}/J_{\lambda'}$, then the ratio $I_\lambda/I_{\lambda'}$ will have been determined.

The problem is thus reduced in principle to the comparison of the light intensities of two spectra at the same wave-length. Or, if we work in terms of magnitudes, given two spectra we wish to determine for the several regions of the spectrum their magnitude differences, measured in each case at the same wave-length. In other words, we wish to determine the quantity $0{\cdot}4 \log (I_\lambda/J_\lambda)$ for various values of $\lambda$.

In determining stellar magnitudes, a well-known method is the use of the coarse diffraction grating. This gives us several images of the same star, differing in magnitude by known amounts, and thus provides us with a magnitude scale on our photographic plate. The extension of this photometric method to spectro-photometric work is a logical step to take, and is in fact what is known as the "prism-crossed-by-grating" method proposed by Professor Hertzsprung.* In essentials it consists in introducing a coarse grating whose dispersion is at right angles to the dispersion of a prism. On a plate we then get in addition to the main central spectrum various flanking spectra, and for any given wave-length the magnitudes of these spectra will differ from the magnitude of the central spectrum by a known amount, depending on the constants of the grating.

Some preliminary work has been carried out at Greenwich on these lines, and it is the object of this note to describe what has been done.

* *Ast. Nach.*, No. 5000, August 1919. A similar method has been used in the laboratory by Prof. T. R. Merton; see *Proc. Roy. Soc.*, April 1921.

The actual figures given below make no claim to be final, but are put forward as illustrative of a method which appears to show some promise.

The instrument employed was the 30-inch Cassegrain reflector, the convex mirror of which was placed so as to send a parallel beam 5 inches in diameter through the central hole of the 30-inch mirror. This beam then passed through a 40° flint prism, and was brought to focus by a 6-inch lens of 27 inches focal length. The grating, made of wires 1·5 mm. in diameter and approximately 1·5 mm. apart, was placed over the mouth of the 30-inch tube with its dispersion in right ascension, the dispersion of the prism being in declination. The dispersion of the central spectrum thus obtained is 11·5 mm. between $H_\beta$ and $H_\epsilon$, and the average distance between the central spectrum and the first order spectra is about 0·7 mm., admitting of a widening of the spectrum bands of half a millimetre without interference. The plates used were Marion Iso-Record plates, which are sensitive up to $\lambda$ 5800, there being, however, a region of low sensitivity in the green.

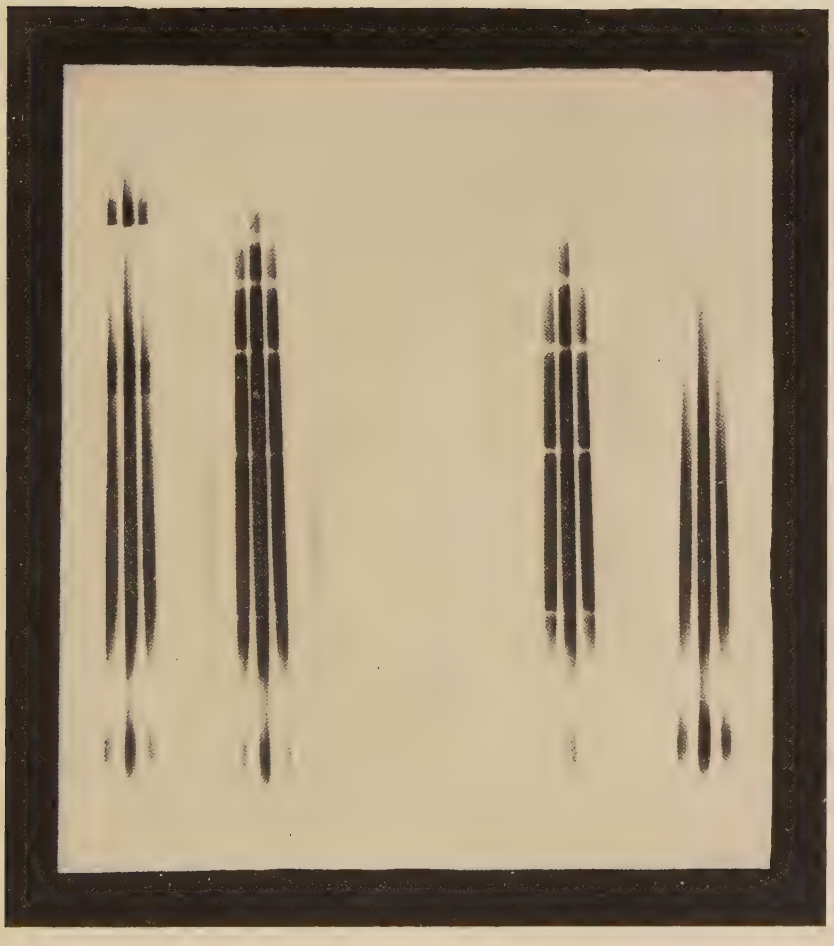
It was originally the intention to employ a carbon arc with " solid " carbons as the source of the comparison spectrum. This was installed on the roof of a building some 600 feet away from the telescope, the position of the convex mirror being appropriately readjusted so as to provide a parallel beam on the prism. By means of a pin-hole an image of the positive crater of the arc was projected on to a screen, in the centre of which was a second hole through which there emerged a beam composed entirely of light from the central portion of the crater. The second hole was large enough to ensure that the telescope was completely filled by the beam.

Unfortunately in use the positive crater is enveloped in an incandescent atmosphere, and superposed on the continuous spectrum of the boiling carbon are cyanogen emission bands, especially towards the violet end of the spectrum. Stellar spectra are also broken, in this case by absorption lines and bands, and it was found difficult to make a direct comparison of the continuous spectra of stars and arc. Accordingly, a half-watt incandescent electric lamp was employed as intermediary, with which both stars and arc were compared.

The accompanying plate shows the spectra of $\alpha$ Lyræ, the arc, and the lamp.

The plates were measured in a micro-photometer constructed at the Royal Observatory. This is of the Hartmann type as modified by Jules Baillaud,* in which by means of a Lummer-Brodhun cube the images of the spectrum and of a graduated wedge, both illuminated by a lamp, are superposed in the focal plane of the viewing telescope. A small stop is placed in the focal plane, and when the eye is brought up to this stop the surface of the Lummer-Brodhun cube is observed illuminated by parallel light, the central zone of which consists of light which has passed through the photographed spectrum, and the outer zone of light which has passed through the wedge. The two zones are brought to equality by means of a movement of the wedge, and a

* *Bulletin de la Société Astronomique de France,* **28**, 28, 1914.




Spectra of α Lyræ, Arc, and Lamp.

measure of the density of photographic deposit of that part of the photographed spectrum under examination is thus obtained.

In comparing the spectrum of a star with that of the lamp, the densities of both the central and first order spectra of both star and lamp were measured at a number of points along the spectra. These measures when reduced give the magnitude differences of the star and lamp for each wave-length measured. The arc and lamp were compared similarly.

The theory of the reduction of the measures will now be given. For a given wave-length $\lambda$ let $D_1$ and $D_2$ be the measured densities of photographic deposit of the central spectrum of a star and of its first order spectra, and let $D_3$ and $D_4$ be the corresponding measures for the lamp. Let $E_1$, $E_2$, $E_3$, $E_4$ denote the logarithms of the corresponding light intensities, divided in each case by 0·4, so that the E's are really magnitudes.

We may assume that the magnitude E is connected with the density of deposit D by a relation of the form

$$E=a+bD+cD^2,$$

where $a$, $b$, and $c$ are constants for the plate and for the wave-length $\lambda$ in question.

We then get the four following relations :—

$$\begin{aligned} E_1&=a+bD_1+cD_1^2,\\ E_2&=a+bD_2+cD_2^2,\\ E_3&=a+bD_3+cD_3^2,\\ E_4&=a+bD_4+cD_4^2. \end{aligned}$$

Furthermore, $E_1-E_2=E_3-E_4=g$, where $g$ denotes the magnitude constant of the grating. For a grating in which bar and space are equal, $g=0{\cdot}98$ magnitudes.

Put

$$\begin{aligned} D_1-D_2&=h, \qquad D_3-D_4=k.\\ D_1+D_2&=p, \qquad D_3+D_4=q. \end{aligned}$$

The above equations then give

$$E_2-E_3=g(D_2-D_3)\left[\tfrac{1}{2}\left(\frac{1}{h}+\frac{1}{k}\right)+\tfrac{1}{2}\frac{(h-k)^2}{hk(p-q)}\right].$$

Experience has shown that the term involving $(h-k)^2$ is negligible, and that the harmonic mean of $h$ and $k$ can be replaced by their arithmetic mean without loss of accuracy.

The formula then becomes

$$E_2-E_3=\frac{2g(D_2-D_3)}{D_1-D_2+D_3-D_4}.$$

Similarly we get

$$E_1-E_4=\frac{2g(D_1-D_4)}{D_3-D_4+D_1-D_2}.$$

Combining these two formulæ and putting $E_\lambda=\frac{1}{2}(E_1+E_2)$ and $E_\lambda'=\frac{1}{2}(E_3+E_4)$ we get

$$E_\lambda-E_\lambda'=\frac{D_1+D_2-D_3-D_4}{D_1-D_2+D_3-D_4}\times g,$$

and this formula was applied to the measures.

Neglecting for the moment any corrections for the effect of selective atmospheric absorption that may have to be applied to the measures, let $T_1$ be the effective temperature of the star and $T_2$ that of the lamp.

Planck's formula then gives

$$0{\cdot}4(E_\lambda-E_\lambda')=A+\log_{10}(e^{x_2}-1)-\log_{10}(e^{x_1}-1),$$

where $x_1=c_2/\lambda T_1$, $x_2=c_2/\lambda T_2$, and A is a constant independent of wave-length.

This gives

$$0{\cdot}4(E_\lambda-E_\lambda')/\log_{10}e=A/\log_{10}e+c_2/\lambda T_2-c_2/\lambda T_1+\log_e(1-e^{-x_2})-\log_e(1-e^{-x_1}),$$

and therefore

$$\frac{d}{d\left(\frac{1}{\lambda}\right)}[E_\lambda-E_\lambda']\times\frac{0{\cdot}4}{\log_{10}e}=c_2/T_2-c_2/T_1+\frac{c_2}{T_2}\frac{1}{e^{x_2}-1}-\frac{c_2}{T_1}\frac{1}{e^{x_1}-1}.$$

If T be measured in degrees on the absolute centigrade scale and $\lambda$ be measured in microns, we have $c_2=14320$. The last term but one on the right-hand side is negligible, and in the last term $\lambda$ may be replaced by its mean value, which is taken to be 0·48.

The formula then becomes

$$c_2/T_2-c_2/T_1=G_1+\frac{c_2}{T_1}\Big/[e^{c_2/0{\cdot}48T_1}-1],$$

where

$$G_1=\frac{d}{d\left(\frac{1}{\lambda}\right)}[E_\lambda-E_\lambda']\times 0{\cdot}922,$$

and is a quantity deduced from the measures.

Similarly if $G_2$ denote the measured gradient for the comparison between arc and lamp, and if $T_3$ denote the temperature of the arc, we have

$$c_2/T_2-c_2/T_3=G_2,$$

the other term being negligible in this case.

Subtracting, we get

$$c_2/T_3-c_2/T_1=G_1-G_2+\frac{c_2}{T_1}\Big/[e^{c_2/0{\cdot}48T_1}-1],$$

giving

$$c_2/T_1=X-\frac{c_2}{T_1}\Big/[e^{c_2/0{\cdot}48T_1}-1]$$
$$=X-r, \text{ say,}$$

where

$$X=c_2/T_3-(G_1-G_2).$$

The values of $r$ as a function of X are given in the following Table :—

| X. | r. | X. | r. | X. | r. |
|---|---|---|---|---|---|
| 5·00 | 0·00 | 1·60 | 0·07 | 1·05 | 0·16 |
| 4·00 | 0·00 | 1·50 | 0·08 | 1·00 | 0·18 |
| 3·00 | 0·00 | 1·40 | 0·09 | 0·95 | 0·20 |
| 2·50 | 0·01 | 1·30 | 0·11 | 0·90 | 0·22 |
| 2·25 | 0·02 | 1·25 | 0·12 | 0·85 | 0·24 |
| 2·00 | 0·03 | 1·20 | 0·13 | 0·80 | 0·25 |
| 1·80 | 0·04 | 1·15 | 0·14 | 0·75 | 0·28 |
| 1·70 | 0·05 | 1·10 | 0·15 | 0·70 | 0·33 |

$c_2/T_1$ can now be derived and hence $T_1$.

The diagram (p. 38) shows for 3 stars, viz. $\alpha$ Lyræ, Polaris, and Arcturus, plots of magnitude difference $E_\lambda - E_\lambda'$ between star and lamp against reciprocal of wave-length. Two plates, each of which had been exposed to these three stars as well as to the arc and lamp, were measured, and the corresponding plots of magnitude difference between arc and lamp are also shown. The plates are distinguished on the diagram by the letters A and B. The dotted portion of the curve for $\alpha$ Lyræ B indicates a region in which the spectra were so faint as to be almost unmeasurable.

For two black-body sources, such a plot of magnitude difference against reciprocal of wave-length should be to a very close approximation a straight line. In the case of the $\alpha$ Lyræ curves this is true as far as can be judged, with the exception of the regions affected by the Balmer series. In the case of Polaris there seems to be a slight curvature. In the Arcturus curves this curvature is clearly manifested, the light from the star being apparently deficient in the blue end of the spectrum. Like all other stars of late type, Arcturus is heavily affected by absorption in this region.

The arc curves exhibit the phenomenon which was referred to previously, the emission due to the incandescent gases surrounding the positive crater being superposed on the continuous radiation emitted by the crater itself. The cyanogen emission in the violet region is clearly shown.

The gradients $G_1$, $G_2$ were derived by adjusting the straight line of closest fit to each of the curves, the regions of Balmer absorption in the case of $\alpha$ Lyræ and the region of cyanogen emission in the case of the arc being ignored. In the case of Arcturus only the parts of the curves of longer wave-length (from $\lambda$ 4500 onwards) were used.

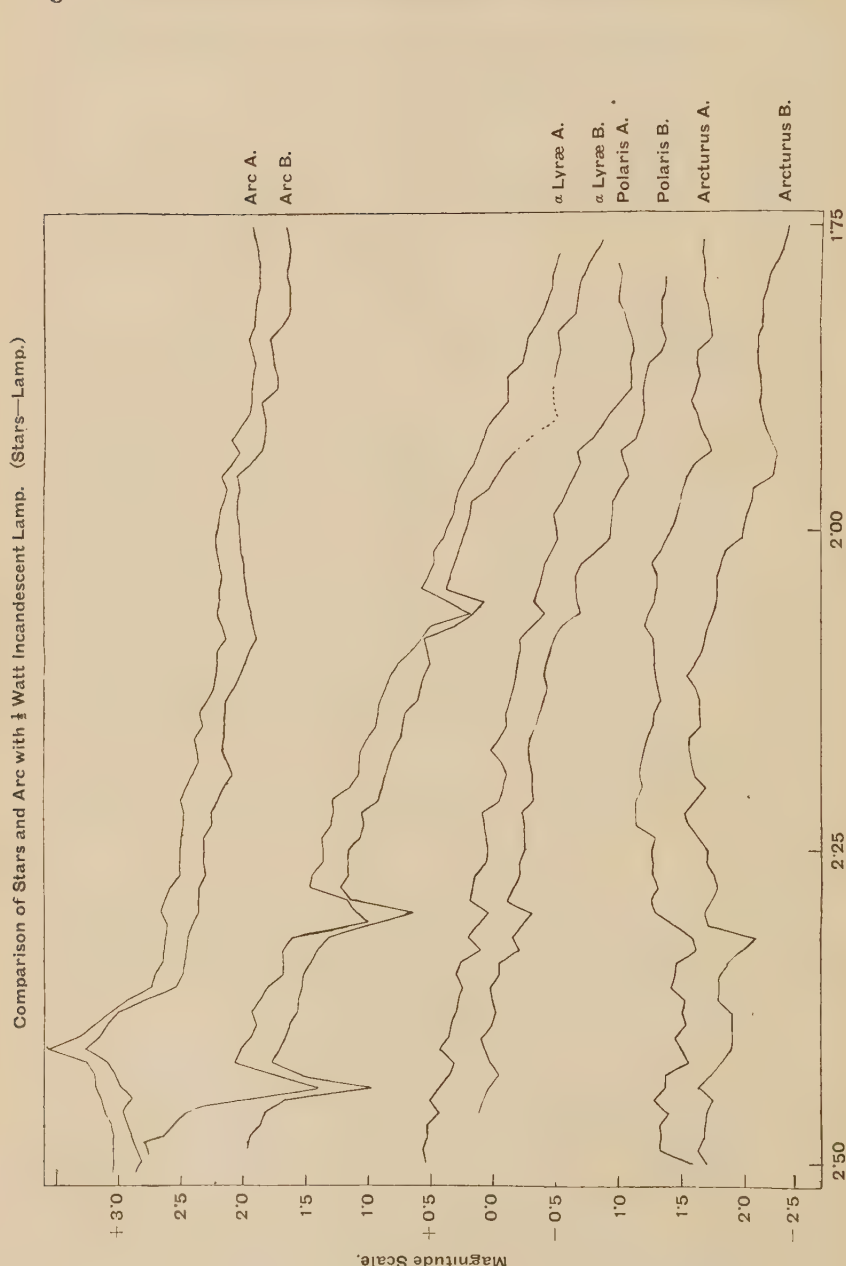
The measured gradients and zenith distances are as follows :—

*Plate A.*

| | | | |
|---|---|---|---|
| $\alpha$ Lyræ —Lamp | Gradient | $G_1=3·69$ | Z.D.=13° |
| Polaris — ,, | ,, | $G_1=2·21$ | Z.D.=38° |
| Arcturus— ,, | ,, | $G_1=1·20$ | Z.D.=44° |
| Arc — ,, | ,, | $G_2=1·34$ | |

*Plate B.*

| | | | |
|---|---|---|---|
| $\alpha$ Lyræ —Lamp | Gradient | $G_1=3·62$ | Z.D.=48° |
| Polaris — ,, | ,, | $G_1=2·11$ | Z.D.=38° |
| Arcturus— ,, | ,, | $G_1=1·62$ | Z.D.=32° |
| Arc — ,, | ,, | $G_2=1·33$ | |


Comparison of Stars and Arc with ½ Watt Incandescent Lamp. (Stars—Lamp.)
Arc A.
Arc B.
α Lyræ A.
α Lyræ B.
Polaris A.
Polaris B.
Arcturus A.
Arcturus B.
+3·0
2·5
2·0
1·5
1·0
+0·5
0·0
−0·5
1·0
1·5
2·0
−2·5
Magnitude Scale.
2·50
2·25
2·00
1·75
Reciprocal of Wave-length (in microns).

For the relative temperatures of the arc and lamp, Plate A gives $c_2/T_2 - c_2/T_3 = 1{\cdot}34$, and Plate B gives $c_2/T_2 - c_2/T_3 = 1{\cdot}33$. Adopting 3800° absolute for the temperature of the arc, these give for the temperature of the lamp 2800° and 2810° absolute.

From the above values the quantities X, defined as above, can be formed. Before deriving temperatures from them they must be corrected for the effects of atmospheric absorption.

Suppose a star be observed at zenith distances $z_1$, $z_2$, and let $X_1$, $X_2$ be the observed gradients. Then, if X be the true gradient, we have

$$X_1 = X + A \sec. z_1,$$
$$X_2 = X + A \sec. z_2,$$

where A is a quantity measured in units of the gradient and may be termed the coefficient of atmospheric absorption.

From these two equations X and A could be determined. The two gradients $X_1$, $X_2$ derived from the above values for $\alpha$ Lyræ are :

$$X_1 = 1{\cdot}42 \qquad z_1 = 13°,$$
$$X_2 = 1{\cdot}48 \qquad z_2 = 48°,$$

giving A = 0·13 and X = 1·29.

This value of X gives for the temperature of $\alpha$ Lyræ, $T_1 = 12100°$.

The above values of A, however, seem to be rather small. Abbot * has given data from which the mean value of A at Washington for the spectral region considered can be determined. It comes out to be 0·32. The conditions at Washington should not be very different from those at Greenwich.

Clearly a good determination of A can only be derived from a large amount of material, and in the present preliminary note it is useless to discuss the matter further. Using the Washington value 0·32, the following are the derived temperatures :—

| | | |
|---|---|---|
| $\alpha$ Lyræ . | Plate A . . | 15200° absolute |
| | Plate B . . | 17500° ,, |
| Polaris . | Plate A . . | 5750° ,, |
| | Plate B . . | 5550° ,, |
| Arcturus | Plate A . . | 4130° ,, |
| | Plate B . . | 4620° ,, |

In the above reductions the temperature of the arc has been taken to be 3800°. Any error in this assumed temperature will appear as a systematic error in the stellar temperatures. Furthermore, since it is the reciprocal of temperature that is necessarily determined, a small error in the assumed temperature of the arc can lead to a much greater error in the temperature of an early type star. For example, if we had taken the temperature of the arc to be 3700°, the deduced temperatures for $\alpha$ Lyræ (using A = 0·32 as before) would have been 13500° for Plate A and 15100° for Plate B.

* *Annals of the Astrophysical Observatory of the Smithsonian Institution*, **2**, 113.

A further possible source of systematic error is the adopted constant of the grating. For a grating in which bar and space are equal $g=0{\cdot}98$ magnitudes. Actually for the Greenwich grating the value $g=0{\cdot}77$ magnitudes was used, corresponding to a ratio of space to space plus bar of 0·444. This value was derived by comparing the densities of the 4th, 5th, and 6th order spectra in the microphotometer. If bar and space had been equal, neither the 4th nor 6th order spectra, nor, in fact, any of the spectra of even order would have appeared.

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*The Solar Motion derived from Radial and from Transverse Velocities of Stars of Spectral Types F to M.* By W. M. H. Greaves, M.A.

(*Communicated by the Astronomer-Royal.*)

It is well known that in the past there has been a discrepancy in the declination of the solar apex as determined from meridian proper motions and from line-of-sight velocities. Kapteyn * has pointed out that the difference (some 10° in declination †) between Boss's determination of the apex ‡ from the proper motions in his Preliminary General Catalogue, and Campbell's determination § from the radial velocities of some 1200 stars, could be explained by supposing that the Boss proper motions need a systematic correction of the form $k\cos\delta$. Kapteyn's value for $k$ is 0″·014 per annum.

Before accepting a correction of such magnitude there must be many who feel that all other conceivable sources of discrepancy should be explored. One possible such source might lie in the fact that Boss's determination is based on stars down to the 6th magnitude and fainter, whereas Campbell's is based on stars down to the 4th magnitude. *A priori* it is doubtful whether the discordance could be explained in this way, but at any rate it seems desirable that determinations of the solar apex should be made from a discussion of the proper motions and radial velocities of *the same selection of stars.*

In Boss's determination the proper motions were analysed by Airy's method. In this method the sky is divided into a number of small areas, and the mean proper motion formed for all the stars in each such area. The method really assumes that the stars can all be regarded as being at the same distance, deviations from this hypothesis being regarded as accidental. A serious drawback to the method lies in the fact that in forming the mean proper motion for each group too much weight is given to the nearer and faster moving stars, and the majority of the stars play only a passive part in the result.

This difficulty could be avoided if the parallaxes of the stars were

* *B.A.N.*, No. 14, April 1922.

† Boss's apex is R.A. 270°·5±1°·5, Dec. +34°·3±1°·3; and Campbell's is R.A. 268°·5±2°·0, Dec. 25°·3±1°·8, with a solar motion of 19·5±0·6 kilometres per second.

‡ *Astronomical Journal*, **26**, No. 14, July 1910.

§ *Lick Observatory Bulletin*, No. 196, 1911.

known and were used to transform the proper motions of the individual stars into linear velocities at the outset. Spectroscopic parallaxes seem to be the most suitable for this purpose in view of the fact that their probable errors are proportional to the parallaxes themselves.

The above considerations led the writer to undertake the determination of the solar apex from those stars in Boss's catalogue for which spectroscopic parallaxes and radial velocities were available. The details of the investigation are given below.

*Material Employed.*—The material consisted of those Boss stars which also appeared in the Mount Wilson Catalogue of 1646 spectroscopic parallaxes,* and for which radial velocity determinations were available. The corrections to the Boss proper motions given on page xxviii of the introduction to the "Preliminary General Catalogue" were applied. The sources from which the radial velocities were drawn were Voûte's first catalogue, *Mount Wilson Contribution*, No. 254, and the Victoria Catalogue of 594 radial velocities.† When radial velocities for any one star were available from more than one of these sources, a simple mean was taken. In the case of double stars for which spectroscopic parallaxes or radial velocities were available for both components, the mean was taken. There were 32 such double stars on the final list. Stars which had been identified as belonging to the Taurus cluster ‡ or the 61 Cygni cluster § were rejected. No star was rejected on account of excessive speed. After the rejection of the cluster stars, and of some others south of declination 18° S. as explained below, the final working list consisted of 916 stars whose declinations all lay between 38° S. and the north pole.

*Method of Analysis.*—Denote the proper motions in Right Ascension and Declination of a star whose Right Ascension is $\alpha$ and Declination $\delta$ by $\mu_\alpha$ and $\mu_\delta$ (measured in seconds of arc per annum), its parallax (measured in seconds of arc) by $\pi$, and its radial velocity (measured in kilometres per second) by $V_r$. Put $V_\alpha=\mu_\alpha/\pi$ and $V_\delta=\mu_\delta/\pi$, so that $V_\alpha$ and $V_\delta$ are velocity components measured in astronomical units per annum. Then if X, Y, Z denote the rectangular components of the reverse of the solar motion measured in kilometres per second and resolved along axes through the equinox, perpendicular to the equinox in the equatorial plane, and through the north pole, each star yields the three following equations of condition :—

$$-X \sin \alpha + Y \cos \alpha = V_\alpha \times 4{\cdot}74 \quad . \quad . \quad . \quad . \quad (1)$$

$$-X \cos \alpha \sin \delta - Y \sin \alpha \sin \delta + Z \cos \delta = V_\delta \times 4{\cdot}74 \quad . \quad (2)$$

$$+X \cos \alpha \cos \delta + Y \sin \alpha \cos \delta + Z \sin \delta + K = V_r \quad . \quad (3)$$

These equations of condition were formed for each of the 916 stars. In order to shorten the labour of reduction the sky was divided into regions, and the equations of condition meaned for each region. The resulting equations were treated as equations of condition, and solutions

* *Mount Wilson Contribution*, No. 199.
† *Publications of the Dominion Astrophysical Observatory*, **2**, No. 1.
‡ L. Boss, *Astronomical Journal*, No. 604.
§ B. Boss, *ibid.*, Nos. 629, 633.

were made by least squares. The grouping into regions proceeded according to every two hours of R.A. and within the following belts of declination: 38° S. to 18° S., 18° S. to the equator, the equator to 18° N., 18° N. to 38° N., and 38° N. to 68° N. The area from declination 68° N. to the north pole was divided into three regions by zones of R.A., viz. $0^h$ to $8^h$, $8^h$ to $16^h$, and $16^h$ to $0^h$. Eight of the twelve regions between declinations 18° S. and 36° S. contained less than six stars each and were rejected, leaving 54 regions for discussion. The average number of stars in one of these 54 regions was 17. In the case of 8 regions, for which the total number of stars in each was less than 12, the corresponding mean equations of condition were half-weighted.

*Results.*

The following are the results for the different solutions made. A and D denote the Right Ascension and Declination of the solar apex, and V the speed of the solar motion in kilometres per second :—

*Solution I.*—From $V_\alpha s$ only [equations (1)].

$$X=-0\cdot2\pm1\cdot1, \qquad Y=+16\cdot8\pm1\cdot1;$$
$$A=270^\circ\cdot7\pm3^\circ\cdot7.$$

*Solution II.*—From $V_\delta s$ only [equations (2)].

$$X=-5\cdot0\pm2\cdot2, \qquad Y=+13\cdot35\pm2\cdot2, \qquad Z=-14\cdot4\pm0\cdot9;$$
$$A=290^\circ\cdot3\pm8^\circ\cdot9, \qquad D=45^\circ\cdot3\pm4^\circ\cdot7,$$
$$V=20\cdot3\pm1\cdot8.$$

*Solution III.*—From $V_\alpha s$ and $V_\delta s$ [equations (1) and (2)].

$$X=-1\cdot0\pm1\cdot0, \qquad Y=+16\cdot05\pm1\cdot0, \qquad Z=-14\cdot3\pm0\cdot9;$$
$$A=273^\circ\cdot6\pm3^\circ\cdot6, \qquad D=41^\circ\cdot7\pm2^\circ\cdot5,$$
$$V=21\cdot5\pm1\cdot0.$$

*Solution IV.*—From $V_r s$ [equations (3)].

$$X=-1\cdot3\pm0\cdot8, \quad Y=+17\cdot3\pm0\cdot8, \quad Z=-10\cdot4\pm1\cdot3, \quad K=+1\cdot3\pm0\cdot7;$$
$$A=274^\circ\cdot3\pm2^\circ\cdot6, \qquad D=30^\circ\cdot9\pm3^\circ\cdot4,$$
$$V=20\cdot2\pm1\cdot0.$$

In the cases of Solutions III. and IV. the residuals were calculated, and from these the probable errors of a group of unit weight were derived. These probable errors came out to be $\pm5\cdot4$ and $\pm3\cdot4$ kilometres per second respectively. The probable errors of X, Y, and Z were then calculated. In the case of Solutions I. and II., the probable errors of X, Y, and Z were derived from the value $\pm5\cdot4$ given by Solution III. as the probable error of a group of unit weight.

The above results show a discrepancy in the values of Z obtained from Solutions III. and IV., which leads to a discrepancy of some 10° in the declination of the apex. This difference is rather larger than is permitted by the probable errors,* and is the same as the difference between the determinations of Boss and Campbell. If, following

* The difference in Z between the two solutions is $3\cdot9\pm1\cdot6$.

Kapteyn, we attribute it to a systematic error of the form $k \cos \delta$ in the $\mu_\delta s$, we find that in order to make Z as derived from Solution III. agree with the Z of Solution IV., the $V_\delta s$ require a systematic correction of $3{\cdot}9 \cos \delta$ km./sec., leading to a systematic correction $+0''{\cdot}022 \cos \delta$ per annum to the $\mu_\delta s$, the average parallax of the stars used being 0·027. Half of this value, however, may easily be accidental.

Although such a correction would make the two solutions agree, it does not follow that the discrepancy between the two solutions implies a systematic correction of the $k \cos \delta$ form to Boss's proper motions. Systematic errors in proper motions in declination may be expected to arise from errors in the assumed refractions and assumed flexures of the meridian instruments, and will, therefore, be functions of the zenith distances at the various observatories, whose observations were used in the compilation of the catalogue. It is then to be expected that the systematic correction to the proper motions will be a more or less complicated function of declination, and will not necessarily be of the simple $k \cos \delta$ type. It can, however, be resolved into a Fourier series of sines and cosines of multiples of $\delta$, and it should be observed that of all the terms in this series, only the term in $\cos \delta$ can affect the derived position of the solar apex, at any rate when the material is distributed over the whole sky. In other words, given an error in the derived position of the apex, and assuming this error to be due to a systematic error in the $\mu_\delta s$, it is impossible to deduce this systematic error, but only one Fourier component of it. The replacement of the entire systematic error by its $k \cos \delta$ Fourier component may be numerically fallacious, and may conceal the physical and instrumental origin of the error.

Again, the possibility that the discrepancy is not due to systematic errors in meridian proper motions, but to some other cause, is one that should not be entirely lost sight of. One of the features of the present investigation is the fact that the various solutions are based on the same selection of stars, and to this extent the results confirm the view that the discrepancy is not due to selection of material. But it is not altogether inconceivable that its origin may be in some peculiarity of the stellar motions.*

It should be observed that although the *difference* between Solutions III. and IV. agrees with the difference between Boss's and Campbell's solution, the values of D in each case are high compared with Boss's and Campbell's values. This is probably due to the fact that the stars used in the present investigation are all of late spectral type.

There is a discordance of $4{\cdot}8 \pm 2{\cdot}5$ kilometres per second between the values of X obtained from Solutions I. and II., leading to a discordance in the Right Ascension of the apex. If some of this discordance is real, it suggests a systematic error in the $V_\delta s$ depending on R.A. This is also suggested by an examination of the residuals.

* Such as, for instance, a variation in the mixture of the two star streams in different parts of the sky. If the ratio of the number of stars in Stream II. to the number of stars in Stream I. varied from 0·5 at the pole to 0·8 at the equator, most of the discrepancy would be accounted for. But Eddington's results (see *Stellar Movements*, p. 117) do not suggest such a variation.

*A Comparison of Schlesinger's Proper Motions of the Harvard A.G. Zone, with Proper Motions Photographically Determined at Cambridge.* By W. M. Smart.

(*Communicated by the Director, Cambridge Observatory.*)

§ 1. At the meeting * of the Society on 1925 January 9, Professor Schlesinger gave a brief account of the photographic repetition, at Yale, of the Harvard zone (+50° to +55°) of the Gesellschaft Catalogue and indicated the order of accuracy of the resulting star-positions. The Catalogue, embodying this new survey of the zone, has now been published.† The positions of the stars for the epoch of observation are given referred to the equinox of 1875·0, and direct comparison with the Harvard Zone Catalogue yields (after the application of certain derived corrections) the proper motions of 8359 stars. It is the object of this note to compare the values of Schlesinger's proper motions with some values recently derived photographically at Cambridge.

Schlesinger's plates were taken with a specially designed doubtlet camera, the objective having an equivalent focal length of 163·5 cm., giving a scale of 126″·2 to the millimetre. The area of the field photographed is 5°×5°. Ninety plates—each overlapping equally the preceding and following plates—cover the zone, and on each plate one exposure of 25 minutes has been given. Accordingly, every star has two determinations of position. The reduction of the plates was effected by means of 1070 selected stars which were specially observed with the meridian circle in 1917–18 by Professor Tucker at the Lick Observatory. There are thus 23 or 24 comparison stars per plate. The positions of these stars have been calculated back to 1875·0, using the precessions tabulated in the Harvard A.G. Catalogue. The measures and reductions of the photographed star images yield, then, the positions of the stars as observed in 1915–16 referred to the epoch of the Harvard Catalogue. The differences in the positions of the stars in this 40-year interval are what Schlesinger calls the "crude" proper motions. The systematic differences of the A.G. Catalogue and of Boss's P.G.C. have been derived by comparing directly the "crude" proper motions and the Boss proper motions of some 143 stars common to the two Catalogues. This systematic correction having been applied, Schlesinger's proper motions are thus reduced—at least approximately—to the Boss system. Finally, from a thoroughly exhaustive examination, Schlesinger concludes that the probable errors of the photographic positions of his Catalogue are ±0″·161 in right ascension and ±0″·156 in declination (roughly a third or a fourth of the meridian probable errors of the Harvard Catalogue). From these and the Harvard probable errors, the probable errors of the proper motions are deduced. These are of the order ±1″·0 per century in right ascension and ±1″·5 per century in declination; the largeness

* Vide *Observatory*, **48**, 35 (February 1925).

† *Transactions of the Astronomical Observatory of Yale University*, **4**.

of these probable errors is due, of course, in most part to the lesser precision of the Harvard meridian observations.

§ 2. Of the score or so of regions measured by us so far at Cambridge there is only one which lies within the Harvard zone. It is Region III. of my fourth paper on the systematic motions of faint stars (*M.N.*, **85**, 433). There are altogether 24 A.G. stars in this region, and the number is large enough for an effective comparison of the proper motions. The Cambridge results are based on an interval of 23 years, with 6 determinations of proper motion for each star. It may be recalled that the proper motions derived first of all from the measures are relative proper motions, *i.e.* relative to the mean motion of the group of reference stars used in the reduction. The absolute proper motions should, in principle, be derived by allowing for a correction given by comparing directly the relative proper motions of a sufficiently representative number of stars with their absolute proper motions based on meridian circle observations. Unfortunately, the number of stars with reliable meridian proper motions is generally inadequate for this purpose, and an alternative method is adopted of calculating the necessary correction to the relative proper motions by means of the parallactic motions of a considerable number of stars of known magnitude. It has been possible for us, only once before, to make a direct comparison of the two methods—on this occasion * 18 meridian proper motions were available for the purpose—and the agreement between the results of the two methods appeared satisfactory. Schlesinger's Catalogue forms, then, a welcome addition to the not too extensive material hitherto at our disposal, and, in particular, affords a further opportunity of testing the results of the "parallactic" method.

§ 3. The table below shows the results for the 24 A.G. stars on our plates. The second and third columns give Schlesinger's values of $\mu_\alpha$ and the probable errors of $\mu_\alpha$, both taken directly from his Catalogue. The fourth column gives our values of $\mu_\alpha$, which are to be considered "absolute" values—the reduction from relative to absolute proper motion having been computed by means of the calculated parallactic motions of 55 B.D. stars in the region photographed. If there is a systematic error in the "parallactic" correction, a statistical comparison of the second and fourth columns should reveal it. In addition, the possibility must not be lost sight of that Schlesinger's precession correction to the "crude" proper motions may not be absolutely accurate (it is derived from 143 Boss stars scattered over the zone); or, stated in another way, that the Boss system of proper motions may have a systematic error which would, likewise, be shown up in the comparisons. The fifth column contains the probable error of the Cambridge proper motions, deduced from the six photographic measures of each star. The sixth column gives the quantity $\Delta\xi$ in the sense:

$$\Delta\xi = \mu_\alpha \text{ (Schlesinger)} - \mu_\alpha \text{ (Smart)}.$$

The remainder of the table gives the corresponding quantities for the proper motions in declination.

* *M.N.*, **84**, 129.

TABLE.

*Comparison of Proper Motions.*

| A.G. | Schlesinger. $\mu_\alpha$. | Schlesinger. P.E. | Smart. $\mu_\alpha$. | Smart. P.E. | $\Delta\xi$. | Schlesinger. $\mu_\delta$. | Schlesinger. P.E. | Smart. $\mu_\delta$. | Smart. P.E. | $\Delta\eta$. |
|---|---|---|---|---|---|---|---|---|---|---|
| | ″ | ″ | ″ | ″ | ″ | ″ | ″ | ″ | ″ | ″ |
| 715 | − 2·2 | ±1·2 | − 0·9 | ±0·1 | −1·3 | −3·0 | ±1·6 | −2·4 | ±0·3 | −0·6 |
| 722 | + 1·5 | 0·9 | + 0·9 | 0·2 | +0·6 | −1·2 | 1·4 | −0·9 | 0·2 | −0·3 |
| 723 | + 1·6 | 1·0 | + 2·3 | 0·2 | −0·7 | 0·0 | 1·3 | +0·5 | 0·1 | −0·5 |
| 731 | − 3·0 | 1·4 | 0·0 | 0·2 | −3·0 | +1·5 | 1·9 | +0·9 | 0·3 | +0·6 |
| 733 | + 2·3 | 1·1 | − 0·4 | 0·2 | +2·7 | +1·4 | 1·4 | −0·9 | 0·2 | +2·3 |
| 734 | + 1·8 | 1·2 | + 1·9 | 0·1 | −0·1 | −1·2 | 1·6 | −1·4 | 0·2 | +0·2 |
| 735 | +15·3 | 1·2 | +14·2 | 0·3 | +1·1 | −7·0 | 1·6 | −6·8 | 0·2 | −0·2 |
| 736 | + 5·2 | 1·0 | + 7·2 | 0·3 | −2·0 | −3·4 | 1·2 | −3·7 | 0·3 | +0·3 |
| 737 | − 0·8 | 1·2 | + 0·8 | 0·2 | −1·6 | +0·4 | 1·7 | −1·3 | 0·2 | +1·7 |
| 742 | + 0·8 | 1·2 | + 0·7 | 0·2 | +0·1 | +0·1 | 1·6 | −1·8 | 0·2 | +1·9 |
| 744 | + 3·8 | 1·1 | + 2·6 | 0·2 | +1·2 | −3·3 | 1·4 | −2·1 | 0·3 | −1·2 |
| 748 | + 1·8 | 1·1 | + 1·0 | 0·2 | +0·8 | −4·1 | 1·6 | −0·7 | 0·1 | −3·4 |
| 749 | + 9·6 | 1·0 | +12·9 | 0·2 | −3·3 | −7·8 | 1·4 | −5·8 | 0·2 | −2·0 |
| 763 | + 2·5 | 1·1 | + 0·9 | 0·2 | +1·6 | +0·9 | 1·6 | −0·5 | 0·1 | +1·4 |
| 766 | 0·6 | 1·1 | − 0·3 | 0·3 | −0·3 | −2·5 | 1·5 | +0·2 | 0·2 | −2·7 |
| 767 | − 1·1 | 1·1 | + 0·8 | 0·4 | −1·9 | +1·7 | 1·6 | −1·9 | 0·2 | +3·6 |
| 768 | − 1·6 | 1·3 | + 2·1 | 0·2 | −3·7 | −4·1 | 1·9 | −1·2 | 0·2 | −2·9 |
| 770 | 0·0 | 1·1 | − 0·6 | 0·2 | −0·6 | +0·6 | 1·6 | 0·0 | 0·2 | +0·6 |
| 772 | + 2·1 | 1·0 | + 0·5 | 0·2 | +1·6 | −1·5 | 1·4 | −2·5 | 0·2 | +1·0 |
| 777 | + 2·5 | 1·0 | + 2·6 | 0·3 | −0·1 | −0·5 | 1·4 | −1·0 | 0·2 | +0·5 |
| 782 | − 1·2 | 0·9 | − 0·3 | 0·2 | −0·9 | −1·7 | 1·1 | −1·6 | 0·1 | −0·1 |
| 786 | − 0·4 | 1·1 | + 1·1 | 0·3 | −1·5 | −1·2 | 1·6 | −0·6 | 0·3 | −0·6 |
| 792 | + 1·9 | 1·0 | + 1·9 | 0·4 | 0·0 | −4·4 | 1·4 | −3·1 | 0·4 | −1·3 |
| 794 | + 0·4 | 1·0 | − 0·2 | 0·3 | +0·6 | +0·2 | 1·3 | +0·2 | 0·2 | 0·0 |

§ 4. A straight mean of the values of $\Delta\xi$—giving unit weight to each of the 24 equations in $\Delta\xi$—yields the result :

$$\Delta\xi = -0''\cdot35 \text{ per century.}$$

The corresponding result for $\Delta\eta$ is :

$$\Delta\eta = -0''\cdot07 \text{ per century.}$$

I have also made the computations, giving to each equation the weight calculated from the probable errors of the table above. The results are practically identical, viz.

$$\Delta\xi = -0''\cdot31 \text{ and } \Delta\eta = -0''\cdot04,$$

the probable errors of these results being negligibly small.

It is thus sufficient to take the values of $\Delta\xi$ and $\Delta\eta$ to be, respectively,

$$-0''\cdot3 \text{ and } 0''\cdot0 \text{ per century.}$$

These values are insignificant, and the general inference is that the parallactic correction and the Boss system of proper motions (in this part of the sky) are sensibly free from systematic error. If we give equal weight to the two methods of deriving the absolute proper motions of the Cambridge stars, we readily infer that the systematic motions * of all the stars in this region remain practically unchanged. It is thus satisfactory to find that the homogeneous material of Schlesinger's Catalogue supports so well the indirect method on which our work at Cambridge on the systematic motions of faint stars is largely based.

§ 5. A further remark of interest—as regards the statistical employment of the proper motions of Schlesinger's Catalogue—is in connection with the relative weights of the proper motions in right ascension and in declination. A casual glance at the Catalogue shows that the probable errors of the proper motions in declination are, on the average, roughly 50 per cent. greater than the probable errors in right ascension —attributable to the assumed higher probable errors in declination of the meridian observations at Harvard. This is not borne out by a comparison of the results for the 24 A.G. stars with which we are concerned in this note. From the second solution in § 4, it is found that the probable error of a single comparison of the proper motions in the table above is, in right ascension, $\pm 1''{\cdot}0$ per century (which is practically Schlesinger's average value of the probable error of $\mu_\alpha$) and $\pm 1''{\cdot}0$ in declination (which is roughly $\frac{2}{3}$ of the average probable error of $\mu_\delta$ in the Catalogue). The evidence, however unimpressive in quantity it may be, appears to indicate that the probable errors of $\mu_\alpha$ as given by Schlesinger are of the right order of magnitude, but that the probable errors of $\mu_\delta$ are too large and should be reduced to about $\frac{2}{3}$ of their tabulated values. If this evidence is characteristic of the whole zone, it would follow that the weights ascribed to the Harvard meridian observations in declination have been somewhat underestimated. In view of the statistical investigations (based on the proper motions of the zone) which are being undertaken at Yale, it would be of very great importance to have more ample confirmation of the greater precision of the values of $\mu_\delta$, as is suggested from the considerations above.

## *Summary.*

§ 6. (*a*) The proper motions of 24 A.G. stars of Schlesinger's Catalogue are compared with the Cambridge photographic values.

(*b*) There is no evidence of any systematic error in the Cambridge proper motions for the region considered, and it is concluded that the "parallactic" method of deriving absolute proper motions from the relative proper motions gives an essentially accurate result.

(*c*) The probable errors of $\mu_\alpha$ in Schlesinger's Catalogue are of the right order of magnitude, but the probable errors of $\mu_\delta$ (as given in the Catalogue) are roughly 50 per cent. too large.

*Observatory, Cambridge*
1925 *November* 4.

* *M.N.*, **85**, 436 (fig. 3).

*A List of Dwarf M Stars.* By Willem J. Luyten.

In view of the interest with which the M dwarfs are generally regarded, since they lie at the dying end of the evolutionary sequence, a tabulation of all observed data concerning them may not be without value. The table below gives, in order, the name, the position for 1900, the Harvard visual magnitude, the spectral class, and the total annual proper motion for 84 stars which, according to the writer's opinion, belong to the group of M dwarfs. It is unfortunate that our present-day classification of spectra does not admit of an absolute criterion, and a list such as given here necessarily rests upon subjective judgment. For example, the star +45° 4408 B, which was given in Adams's and Joy's list of M dwarfs, has not been included in the present list since, according to the writer, it is of no later spectral class than 61 Cygni B. No attempt at an accurate classification has been made here; Ma has been made Mo, and Mb is simply given as M3, without further subdivisions. The last three columns contain the trigonometric parallax, the corresponding absolute magnitude, and the radial velocity in km./sec., when known.

Analysis of the data shows that the mean absolute magnitude of the dwarfs of class Mo is 9·7 (standard deviation 0·8), and that of class M3 is 11·2.

| Star. | R.A. h m | Dec. ° ′ | m. | Sp. | μ. ″ | π. ″ | M. | V. |
|---|---|---|---|---|---|---|---|---|
| +43 44 A | 0 12·7 | +43 27 | 8·1 | Mo | 2·89 | 0·280 | 10·3 | + 5 |
| +43 44 B | 12·7 | +43 27 | 10·7 | M3 | 2·89 | 0·280 | 12·9 | + 5 |
| +66 34 | 26·2 | +66 42 | 9·7 | Mo | 1·77 | | | |
| +33 99 | 38·2 | +33 18 | 9·0 | Mo | 0·43 | | | |
| Wolf 33 | 45·3 | +57 45 | 11·0 | Mo | 1·58 | 0·056 | 9·7 | |
| +61 195 | 56·3 | +61 48 | 11·0 | Mo | 0·72 | | | |
| + 2 348 | 2 07·4 | + 3 10 | 10·5 | Mo | 2·60 | | | |
| + 6 398 B | 30·8 | + 6 25 | 11·5 | M3 | 2·32 | 0·144 | 12·3 | + 24 |
| + 2 418 | 37·3 | + 2 57 | 10·2 | Mo | 0·21 | 0·035 | 7·9 | − 13 |
| | 3 02·8 | +45 22 | 10·2 | Mo | 0·52 | | | |
| $o_2$ Eri. C | 4 10·7 | − 7 49 | 10·8 | M3e | 4·08 | 0·198 | 12·3 | − 42 |
| Kapt. Star | 5 07·7 | −44 59 | 9·0 | Mo | 8·76 | 0·309 | 11·4 | +242 |
| α Aur. C | 10·0 | +45 44 | 9·8 | Mo | 0·44 | 0·069 | 9·0 | + 30 |
| − 3 1123 | 26·4 | − 3 42 | 8·8 | Mo | 2·22 | 0·171 | 10·0 | + 11 |
| +17 1320 | 6 31·4 | +17 38 | 9·7 | Mo | 0·88 | 0·103 | 9·8 | |
| Wolf 294 | 48·4 | +33 24 | 10·0 | Mo | 0·84 | | | |
| α Gem. C | 7 28·2 | +32 08 | 9·4 | Moe | 0·20 | 0·081 | 8·9 | + 2 |
| −11 2741 | 9 46·2 | −11 49 | 9·3 | Mo | 1·85 | | | |
| +63 869 | 48·8 | +63 17 | 8·5 | Mo | 0·60 | | | |
| Wolf 335 | 55·1 | +32 49 | 11·2 | M | 1·20 | | | |
| +20 2465 | 10 14·2 | +20 22 | 9·2 | Moe | 0·49 | 0·205 | 10·8 | + 10 |

| Star. | R.A | Dec. | m. | Sp. | $\mu$. | $\pi$. | M. | V. |
|---|---|---|---|---|---|---|---|---|
| | h m | ° ′ | | | ″ | ″ | | |
| +36 2147 | | 57·9+36 38 | 7·6 | Mo | 4·78 | 0·409 | 10·7 | − 88 |
| +44 2051 | 11 | 00·5+44 02 | 8·9 | Mo | 4·52 | 0·185 | 10·2 | + 65 |
| +31 2240 B | | 05·6+31 00 | 9·8 | Mo | 0·62 | 0·082 | 9·4 | − 18 |
| +66 717 | | 14·8+66 23 | 9·2 | Mo | 2·99 | 0·122 | 9·6 | + 47 |
| Wolf 386 | | 18·6+ 9 07 | 10·5 | Mo | 1·05 | | | |
| −65 1714 A | | 42·2−65 12 | 10· | M | 0·55 | | | |
| −65 1714 B | | 42·2−65 12 | 11· | M | 0·55 | | | |
| +29 2279 | 12 | 14·5+28 56 | 9·5 | Mo | 0·64 | | | |
| +42 2296 | | 16·9+42 42 | 9·1 | Mo | 0·57 | | | + 15 |
| + 9 2636 | | 26·3+ 9 23 | 8·8 | Mo | 0·96 | | | |
| + 0 2989 | | 45·6− 0 13 | 8·7 | Mo | 0·39 | | | |
| +35 2436 | 13 | 14·9+35 39 | 9·6 | Mo | 0·88 | 0·080 | 9·1 | − 2 |
| +18 2776 | | 40·2+18 20 | 9·6 | Mo | 1·86 | 0·082 | 9·2 | + 29 |
| +15 2620 | | 40·7+15 26 | 8·5 | M3 | 2·30 | 0·218 | 10·2 | + 15 |
| +47 2112 | | 58·5+46 49 | 8·8 | Mo | 0·61 | | | |
| +81 465 | 14 | 10·4+81 05 | 9·5 | Mo | 0·58 | 0·062 | 8·5 | |
| +24 2733 A | | 21·1+24 06 | 9·5 | Mo | 1·39 | 0·060 | 8·4 | |
| +24 2733 B | | 21·1+24 06 | 9·6 | Mo | 1·40 | 0·060 | 8·5 | |
| Proxima | | 22·9−62 15 | 10·5 | M | 3·90 | 0·800 | 15·0 | − 22 |
| − 7 3856 | | 25·6− 8 12 | 9·3 | Mo | 1·26 | | | |
| −20 4123 | | 51·6−20 58 | 8·9 | Mo | 2·04 | 0·190 | 10·3 | + 22 |
| | | 52·0+72 05 | 10·8 | Mo | 0·92 | 0·022 | | |
| − 7 4003 | 15 | 14·2− 7 21 | 9·2 | Mo | 1·33 | | | |
| +67 935 B | 16 | 16·6+67 30 | 10·6 | Mo | 0·50 | 0·100 | 10·6 | − 14 |
| | | 21·1+48 36 | 10·7 | M3 | 1·23 | 0·133 | 11·3 | |
| −12 4523 | | 24·8−12 18 | 9·5 | Mo | 1·24 | | | |
| | | 49·6− 8 08 | 11·4 | M | 1·23 | 0·160 | 12·4 | + 12 |
| − 8 4352 | | 50·1− 8 09 | 9·2 | Moe | 1·23 | 0·160 | 10·2 | + 12 |
| +25 3173 | | 54·1+25 55 | 9·4 | Mo | 0·52 | | | |
| +42 2810 | 17 | 09·9+42 28 | 9·4 | Mo | 1·06 | 0·040 | | |
| | | 16·7+41 50 | 10·6 | Mo | 0·88 | | | |
| | | 33·4+18 37 | 9·1 | Mo | 1·39 | 0·126 | 9·6 | 9 |
| 26 Dra. C | | 34·5+61 45 | 10·5 | Mo | 0·57 | 0·071 | 9·8 | − 12 |
| +68 946 | | 37·0+68 26 | 9·5 | M3 | 1·33 | 0·236 | 11·4 | + 39 |
| +43 2796 | | 40·9+43 26 | 9·5 | Mo | 0·62 | | | |
| +27 2888 B | | 42·5+27 47 | 9·7 | M3 | 0·82 | 0·107 | 9·9 | − 16 |
| +27 2888 C | | 42·5+27 47 | 9·8 | M3 | 0·82 | 0·107 | 10·0 | − 16 |
| Barn. Star | | 52·9+ 4 25 | 9·7 | M3 | 10·25 | 0·540 | 13·4 | −117 |
| −43 8486 | 18 | 05·1−43 27 | 8·1 | Mo | 0·49 | | | |
| +59 1915 A | | 41·7+59 29 | 8·8 | M3 | 2·31 | 0·290 | 10·1 | + 4 |

| Star. | | R.A. h m | Dec. ° ′ | m. | Sp. | μ. ″ | π. ″ | M. | V. |
|---|---|---|---|---|---|---|---|---|---|
| +59 | 1915 B | 41·7 | +59 29 | 9·4 | M3 | 2·31 | 0·290 | 10·7 | + 4 |
| + 5 | 3993 | 53·1 | + 5 48 | 9·8 | Mo | 1·25 | 0·091 | 9·6 | + 19 |
| | | 19 03·7 | +32 21 | 11·3 | Mo | 1·66 | 0·120 | 11·7 | |
| + 4 | 4048 | 12·1 | + 5 03 | 9·5 | Mo | 1·51 | 0·117 | 9·8 | |
| Wolf | 1130 | 20 02·8 | +54 10 | 11·4 | M | 1·48 | | | |
| | | 41·5 | +44 08 | 9·4 | Mo | 0·50 | | | |
| +61 | 2068 | 51·3 | +61 48 | 8·6 | Mo | 0·77 | 0·144 | 9·4 | − 8 |
| Wolf | 901 | 54·1 | + 3 11 | 11·4 | M | 1·11 | | | |
| | | 56·2 | +39 41 | 9·7 | M3e | 0·67 | 0·080 | 9·2 | |
| −39 | 8920 | 21 11·4 | −39 15 | 6·6 | Mo | 3·53 | 0·240 | 8·5 | + 13 |
| −49 | 11439 | 26·9 | −49 42 | 8·6 | Mo | 0·82 | | | |
| +27 | 4120 | 33·5 | +27 16 | 10·4 | Mo | 0·50 | | | |
| −58 | 7893 | 39·4 | −58 08 | 8·6 | Mo | 0·90 | | | |
| | | 49·8 | +41 19 | 10·1 | Mo | 0·53 | | | |
| +56 | 2783 A | 22 24·4 | +57 12 | 9·6 | M3 | 0·87 | 0·263 | 11·7 | − 24 |
| +43 | 4305 | 42·5 | +43 50 | 9·5 | Moe | 0·86 | | | |
| −36 | 9694 | 59·4 | −36 26 | 7·4 | Mo | 6·90 | 0·280 | 9·6 | + 12 |
| Wolf | 1039 | 23 29·2 | − 0 21 | 11·4 | Mo | 1·44 | 0·059 | 10·3 | |
| | | 30·8 | +41 25 | 10·8 | Mo | 0·72 | | | |
| + 1 | 4774 | 44·0 | + 1 52 | 8·7 | Mo | 1·39 | 0·170 | 9·8 | − 62 |
| +45 | 4378 | 53·5 | +46 10 | 9·2 | Mo | 0·64 | | | + 5 |
| −37 | 9435 | 59·5 | −37 51 | 8·3 | Mo | 6·11 | 0·220 | 10·0 | + 26 |
| | | 59·9 | +45 14 | 10·2 | Mo | 0·86 | 0·100 | 10·2 | |
| +18 | 683 | 4 37·0 | +18 47 | 10·0 | Mo | 1·26 | | | |

*Harvard College Observatory:*
1925 *September.*

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## *The Rousdon Observations of S Cassiopeiæ* (1901–1920).

By A. Burnet and H. H. Turner.

1. *Introductory Note by H. H. T.*—The Rousdon observations to 1900 are printed in full detail in *Mem. R.A.S.*, **55**. On Mr. Grover's death (1921 Feb. 16) the observations he had made assiduously (up to the day before his death) were put into my hands for editing. The series of observations by the same observer with the same instrument is a long one (1885–1920), and fortunately what can now be done to extend it is being done by the kind transference of the telescope to the Durham Observatory, where Mr. F. Sargent is continuing the observations on as nearly as possible the same plan. But the termination of the actual Rousdon observations clearly indicates a suitable moment for publication.

2. Two circumstances have somewhat delayed the publication: one the great cost of printing since the war; the other an oscillation

in decision as to the best form of presentation. In vol. lx. the Rousdon practice of using the negative sign to denote "fainter than" was retained; but in dealing with Pogson's observations in vol. lviii. the advantages of using the positive sign were so impressive that after some hesitation I decided to adopt the Pogson plan for the remaining Rousdon observations, and they were all written out in that way. Pogson uses the adopted magnitudes of the comparison stars, expressed in units of 0·1, to denote them, so that his comparisons are written

$$101-5=96,$$

meaning that the variable is 0·5 mag. brighter than a comparison star of adopted magnitude 10·1, and therefore is itself of magnitude 9·6. The Rousdon plan is to write 10·1+0·5 for the same comparison. It is not difficult to invert the sign mentally, but it is simpler to invert it at once. However, the Pogson plan has not been adopted, and is apparently repugnant to variable star observers; and these reasons, added to the admitted undesirability of changing the plan in mid-operation, checked the publication of the results written out in Pogson's notation.

3. The plan of publication which now seems to me best is this. Instead of reversing the signs once more, publish only the deduced magnitudes and place the detailed observations (written out on Pogson's plan) in the R.A.S. library, where they can be referred to if necessary. These individual observations are no longer so important, seeing that their general characteristics were fully discussed in vol. lv., and it has been ascertained that the one most likely to change (Mr. Grover's personal peculiarity) remained sensibly constant (see *M.N.*, **85**, 1010). Moreover, during these later years observers have been much more numerous, so that the claims of any series to appear in detail are on a very different footing. If in the future it should appear that the claims of these particular observations to publication are still strong, the material will be accessible. Meanwhile the important results (such as changes in period) will be rendered accessible. A beginning is therefore made with S Cassiopeiæ. In the comparison of the observations with a light curve and deduction of maxima, I have had the kind and efficient help of Mr. A. Burnet, formerly at the University Observatory, and now at the Hepple Woodside Observatory, Northumberland.

*S Cassiopeiæ.*

R.A. 1h. 12m. 18s. Dec.+72° 5′·1 (1900·0).

4. The concluded magnitudes are given in Table I. in units of 0·1 mag. They have been corrected for "stars" and "scale" by the rules adopted in *Mem. R.A.S.*, vol. lv., for the former Rousdon observations. On p. lxix of that memoir the mean results are collected in groups, and exhibited as corrections to a light curve. The corresponding information for the later observations is given in Table II.: it is clear that the adopted period of 624 days is no longer applicable, and Mr. Burnet used both a different period and a different light curve for discussion of the present observations.

TABLE I.

*Magnitudes (Unit* 0·1*) inferred from Rousdon Observations,* 1901–1920.

| J.D. | Mag. | J.D. | Mag. | J.D. | Mag. | J.D. | Mag. | J.D. | Mag. | J.D. | Mag. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5422 | 124 | 6174 | 110 | 7036 | 99 | 8046 | 104 | 9043 | [120 | 955 | [120 |
| 5454 | (130) | 6210 | 98 | 7055 | 103 | 8060 | 96 | 9246 | [120 | 1002 | [120 |
| 5495 | 119 | 6221 | 89 | 7088 | 103 | 8072 | 89 | 9294 | 91 | 1009 | 114 |
| 5522 | 99 | 6242 | 84 | 7100 | 104 | 8079 | 87 | 9300 | 82 | 1012 | 117 |
| 5527 | 99 | 6260 | 83 | 7105 | 105 | 8083 | 90 | 9490 | 102 | 1017 | 117 |
| 5546 | 95 | 6293 | 83 | 7135 | 110 | 8092 | 85 | 9504 | 105 | 1075 | 105 |
| 5570 | 100 | 6304 | 83 | 7240 | [130 | 8105 | 88 | 9539 | 109 | 1096 | 104 |
| 5582 | 95 | 6328 | 85 | 7269 | [130 | 8110 | 91 | 9847 | | 1285 | 96 |
| 5597 | 94 | 6342 | 85 | 7289 | [130 | 8114 | 87 | 9908 | [120 | 1297 | 100 |
| 5605 | 91 | 6367 | 91 | 7316 | 108 | 8118 | 88 | 9913 | 90 | 1311 | 104 |
| 5618 | 86 | 6375 | 94 | 7324 | 107 | 8122 | 88 | 9919 | 89 | 1338 | 104 |
| 5633 | 82 | 6389 | 94 | 7388 | [100 | 8136 | 92 | 9946 | 88 | 1374 | 104 |
| 5649 | 81 | 6397 | 101 | 7416 | 95 | 8141 | 88 | 9971 | 88 | 1402 | 106 |
| 5657 | 82 | 6405 | 98 | 7423 | 91 | 8151 | 91 | 9977 | 88 | 1425 | 109 |
| 5667 | 82 | 6455 | 105 | 7437 | 89 | 8162 | 91 | 0007 | 91 | 1661 | [120 |
| 5704 | 85 | 6493 | 111 | 7450 | 88 | 8171 | 91 | 0019 | 93 | 1722 | [120 |
| 5766 | 100 | 6550 | 112 | 7461 | 84 | 8191 | 95 | 0041 | 99 | 1743 | 104 |
| 5832 | 107 | 6586 | [130 | 7470 | 81 | 8217 | 99 | 0057 | 99 | 1749 | 99 |
| 5846 | 107 | 6606 | 130 | 7480 | 79 | 8252 | 104 | 0198 | [110 | 1762 | 86 |
| 5868 | 111 | 6619 | [130 | 7496 | 83 | 8271 | 100 | 0240 | [115 | 1771 | 83 |
| 5877 | [120 | 6653 | 130 | 7514 | 82 | 8380 | [130 | 0269 | 113 | 1804 | 88 |
| 5889 | 121 | 6694 | 130 | 7546 | 90 | 8431 | [130 | 0305 | [115 | 2046 | [110 |
| 5908 | [125 | 6708 | 107 | 7559 | 88 | 8552 | 107 | 0324 | [110 | 2158 | [115 |
| 5926 | [125 | 6722 | 109 | 7637 | 100 | 8569 | 108 | 0549 | 84 | 2395 | 80 |
| 5945 | [130 | 6730 | 111 | 7646 | 103 | 8588 | 105 | 0558 | 85 | 2420 | 79 |
| 5958 | [130 | 6741 | 110 | 7660 | 102 | 8726 | 86 | 595 | 86 | 2436 | 85 |
| 5970 | [130 | 6753 | 108 | 7673 | 103 | 8729 | 85 | 602 | 87 | 2458 | 87 |
| 5987 | 112 | 6761 | 108 | 7690 | 107 | 8735 | 87 | 611 | 90 | 2490 | 91 |
| 5998 | 116 | 6799 | 106 | 7803 | [120 | 8746 | 86 | 616 | 90 | 2524 | 95 |
| 6006 | 130 | 6829 | 104 | 7812 | [120 | 8757 | 89 | 626 | 87 | 2546 | 102 |
| 6034 | [130 | 6872 | 86 | 7828 | 116 | 8778 | 91 | 649 | 94 | 2557 | 102 |
| 6049 | 130 | 6903 | 88 | 7856 | [130 | 8894 | 104 | 663 | 96 | | |
| 6088 | 130 | 6971 | 91 | 7873 | [130 | 8899 | 104 | 688 | 100 | | |
| 6148 | 113 | 6984 | 91 | 8020 | [130 | 8916 | 107 | 941 | [120 | | |
| 6163 | 109 | 7019 | 97 | 8036 | 107 | 8941 | 109 | 947 | 115 | | |

TABLE II.

(The unit is 0·1 magnitude.)

| Group. | | 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. | 10. | 11. | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| J.D. | E. | 0. | 52. | 104. | 156. | 208. | 260. | 312. | 364. | 416. | 468. | 520. | 572. |
| 5323 | 23 | .. | .. | 0 | .. | − 7 | −2 | 0 | − 2 | −3 | + 2 | −4 | − 5 |
| 5947 | 24 | −10 | .. | .. | + 2 | + 3 | −1 | − 1 | − 4 | +1 | − 5 | −8 | −12 |
| 6571 | 25 | .. | .. | −14 | − 6 | .. | +4 | + 1 | + 5 | −1 | −21 | −8 | + 8 |
| 7195 | 26 | .. | .. | −15 | .. | −12 | −8 | + 3 | + 2 | +2 | − 2 | .. | .. |
| 7819 | 27 | − 8 | .. | .. | .. | − 5 | 0 | + 7 | + 8 | +4 | .. | .. | .. |
| 8443 | 28 | .. | .. | −16 | .. | .. | −2 | + 7 | .. | +4 | + 2 | .. | .. |
| 9067 | 29 | .. | .. | .. | .. | −18 | .. | .. | .. | +8 | + 5 | .. | .. |
| 9691 | 30 | .. | .. | .. | .. | −14 | +1 | +12 | +13 | .. | .. | .. | .. |
| 0315 | 31 | .. | .. | .. | .. | −18 | −1 | +12 | +13 | .. | .. | .. | .. |
| 0939 | 32 | .. | −8 | −16 | −15 | .. | .. | + 3 | +15 | +6 | + 2 | .. | .. |
| 1563 | 33 | .. | .. | .. | −17 | −20 | .. | .. | .. | .. | .. | .. | .. |
| 2187 | 34 | .. | .. | .. | .. | −23 | +3 | +14 | +15 | | | | |
| Mean | | − 9 | −8 | −12 | − 9 | −13 | −1 | + 6 | + 7 | +3 | − 2 | −7 | − 3 |
| Mag. from curve | | 124 | 124 | 124 | 119 | 110 | 96 | 84 | 86 | 93 | 103 | 114 | 123 |

5. Instead of using such information (when adapted to a more suitable light curve) to find merely the period (by taking differences between groups) the correction to epoch was determined for each cycle separately, for which very slight modification in the procedure is needed. When the correction as at first found seemed large, a new determination was made by shifting the light curve by the amount approximately indicated. The ultimate dates of maximum are given in Table III., and the similar maxima for the earlier observations are added, deduced from Table XXIV. of vol. lv. Further, the maxima collected by Dr. Annie Cannon in *H.A.*, **55**, part ii., p. 116, are given in the third column, taking for each value of E the simple mean of separate results when more than one is given.

6. In $C_1$ a series of maxima with uniform period 614 days is shown, and it is seen from the column R in $O-C_1$ that this suits the later Rousdon observations, from which indeed it was deduced.

7. But a different period seems to be indicated by the Harvard maxima, which are positive for E=0 to 7 and negative from E=12 to 19. If we assume that these departures are due to a difference in period, we can reduce them considerably by adopting the period 605 days as in the column $C_2$. But we must then have a change to 614 days after E=19, accompanied by a "jump" in epoch. It will be seen, however, that the period 605 days does not quite suit the observations E=0 to 7, which now show a distinct increase. They were satisfactorily steady with the period 614 days, and it seems possible that the positive residuals in $O-C_1$ for E=0 to 7 represent a systematic habit of the older observers, while the negative residuals from E=15 to 19 represent

another systematic habit of the new set of observers after the break. For instance, the telescopes would tend to be larger after the break from E=7 to E=12.

TABLE III.

*Maxima of S Cassiopeiæ.*

| E. | Rousdon. | H.A. lv. | $C_1$. | O–$C_1$. R. | O–$C_1$. H.A. | $C_2$. | O–$C_2$. R. | O–$C_2$. H.A. |
|---|---|---|---|---|---|---|---|---|
| 0 | .. | 1613 | 1566 | .. | +47 | 1650 | .. | − 37 |
| 2 | .. | 2837 | 2794 | .. | +43 | 2860 | .. | − 23 |
| 3 | .. | 3465 | 3408 | .. | +57 | 3465 | .. | 0 |
| 4 | .. | 4087 | 4022 | .. | +65 | 4070 | .. | + 17 |
| 5 | .. | 4691 | 4636 | .. | +55 | 4675 | .. | + 16 |
| 7 | .. | 5907 | 5864 | .. | +43 | 5885 | .. | + 22 |
| 12 | .. | 8916 | 8934 | .. | −18 | 8910 | .. | + 6 |
| 13 | .. | 9504 | 9548 | .. | −44 | 9515 | .. | − 11 |
| 14 | .. | 0110 | 0162 | .. | −52 | 0120 | .. | − 10 |
| 15 | 0715 | 0743 | 0776 | −61 | −33 | 0725 | − 10 | + 18 |
| 16 | 1306 | 1322 | 1390 | −84 | −68 | 1330 | − 24 | − 8 |
| 17 | 1950 | 1936 | 2004 | −54 | −68 | 1935 | + 15 | + 1 |
| 18 | 2552 | 2548 | 2618 | −66 | −70 | 2540 | + 12 | + 8 |
| 19 | 3175 | 3174 | 3232 | −57 | −58 | 3145 | + 30 | + 29 |
| 20 | 3832 | 3836 | 3846 | −14 | −10 | 3750 | + 82 | + 86 |
| 21 | 4460 | 4462 | 4460 | 0 | + 2 | 4355 | +105 | +107 |
| 22 | 5085 | 5078 | 5074 | +11 | + 4 | 4960 | +125 | +118 |
| 23 | 5680 | 5651 | 5688 | − 8 | −37 | 5565 | +115 | + 86 |
| 24 | 6316 | 6271 | 6302 | +14 | −31 | 6170 | +156 | +101 |
| 25 | 6948 | 6915 | 6916 | +32 | − 1 | 6775 | +193 | +140 |
| 26 | 7521 | .. | 7530 | − 9 | | | | |
| 27 | 8156 | .. | 8144 | +12 | | | | |
| 28 | 8784 | .. | 8758 | +26 | | | | |
| 29 | 9370 | .. | 9372 | − 2 | | | | |
| 30 | 9973 | .. | 9986 | −13 | | | | |
| 31 | 0580 | .. | 0600 | −20 | | | | |
| 32 | 1235 | .. | 1214 | +21 | | | | |
| 33 | 1850 | .. | 1828 | +22 | | | | |
| 34 | 2439 | .. | 2442 | − 3 | | | | |

8. If, however, we accept a change in period, it seems clear that it must be an *increase*.

9. Now S Cassiopeiæ is assigned by Phillips to Group II. (*B.A.A. Journal*, 1916 Oct. 25). There seems to be no room for doubt about this assignment as Phillips finds $\phi_2=135°$ $\phi_3=107°$. Hence the star

goes directly contrary to the suggestion that the periods of stars in Group II. are decreasing (*M.N.*, **80**, pp. 57–64, pp. 274–289, and pp. 481–495), if it shows a large *increase* of period.

10. In a recent paper (*M.N.*, **85**, p. 811) Dr. Jeans considers the possibilities of the two groups from a theoretical point of view. In his verbal account of his paper (*Observatory*, 1925 July, p. 211) he said :

> " I have not so far been able to use Professor Turner's suggestion of development, through first a decreasing and then an increasing period—I rather want the period to decrease all the time."

To which the reply was made by Professor Turner :

> " I admit that it is much easier to establish the shortening in one group than the lengthening in the other. But I will look at the work again in the light of the President's paper."

11. It is rather curious that the first opportunity for recurring to the subject should provide a case of possible *increase* of period which suits neither Dr. Jeans nor Professor Turner. There is just one possibility of converting it to both uses.* The evidence for the smaller period from E=0 to E=18 has a weak point in the gap between E=7 and E=12 ; in which it is conceivable that some change took place. There is also the possibility that the values of E before the gap should be increased by 1 : and we may examine this first. The mean maxima for E=0 to 7 and E=12 to 17 may be put at

| | | | |
|---|---|---|---|
| E= 3·5 | | 2413767 | |
| | 11·0 | | 6655 |
| E=14·5 | | 2420422 | |

To divide by 11 gives the period 605 days already used. If we divide by 10 we get 665·5 days, providing a *decrease* of period to 614 days instead of an *increase*. The individual maxima would then stand as follows :—

Table IV.

| E | O. | C | O-C. | E. | O. | C. | O-C. |
|---|---|---|---|---|---|---|---|
| 1 | 1613 | 1438 | +175 | 12 | 8916 | 8759 | +157 |
| 3 | 2837 | 2769 | + 68 | 13 | 9504 | 9424 | + 80 |
| 4 | 3465 | 3435 | + 30 | 14 | 0110 | 0090 | + 20 |
| 5 | 4087 | 4100 | − 13 | 15 | 0743 | 0755 | − 12 |
| 6 | 4691 | 4766 | − 75 | 16 | 1322 | 1421 | − 99 |
| 8 | 5907 | 6097 | −190 | 17 | 1936 | 2086 | −150 |

12. Clearly we cannot accept this as a possibility. Taking the other alternative, if there is some unknown change (a " jump " in the maximum such as has often been noticed in other cases), then we can only determine the earlier period from the maxima E=0 to 7. The residuals $O-C_2$ prepare us for finding a longer period than that adopted in $C_2$. Treating them by the usual method

* It appears that Dr. Jeans was thinking of changes of period at much greater intervals, so that these remarks lose their point. (See report of the November meeting in the *Observatory*.)

| E. | | Diff. | O-$C_2$. | | Diff. | | Prod. | Squares. |
|---|---|---|---|---|---|---|---|---|
| 0 | 7 | +7 | −37 | +22 | +59×7 | = | 413 | 49 |
| 2 | 5 | +3 | −23 | +16 | +39×3 | = | 117 | 9 |
| 3 | 4 | +1 | 0 | +17 | +17×1 | = | 17 | 1 |
| | | | | | | | 547 | 59 |

we get a correction of +9·3 days to the adopted period of 605·0 days, making it 614·3 days. This is indeed slightly larger than the 614·0 days adopted after E−19, but the difference is too small to be taken seriously. The most favourable hypothesis we can make is that of no change of period within the years of observation.

13. One further point is worthy of examination. There may have been some fault in the light curve or in its expression in harmonics. To test this the observations of Table I. were all referred to the uniform period 614 days and means of every consecutive 5 of them were taken (see Table V.). From these a smooth curve was readily drawn for 420 days out of the 614, the observations near minimum being entirely ignored. But it was found that 9 points of the 12 required for the light curve were thus given directly, and the remaining 3 could be assigned within very narrow limits. They are bracketed in the set given below :

114, 107, 99, 84, 87, 92, 99, 106, 110, (113), (116), (116),

and solving these by harmonic analysis we get

$$14{\cdot}8\cos(\theta-301^\circ)+3{\cdot}8\cos(2\theta-25^\circ)+1{\cdot}3\cos(3\theta-113^\circ)$$

or

$$14{\cdot}8\cos(\theta-180^\circ)+3{\cdot}8\cos(2\theta-143^\circ)+1{\cdot}3\cos(3\theta-107^\circ),$$

in which last expression $\phi_2=143^\circ$ is almost, and $\phi_3=107^\circ$ is exactly, the same as found by Phillips. It may be of interest to give the whole set of means of 5 consecutive observations arranged in 614 days.

TABLE V.

| Day. | Mag. | Day. | Mag. | Day. | Mag. | Day. | Mag. | Day. | Mag. | Day. | Mag. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 78 | 112 | 178 | 99 | 229 | 84 | 273 | 85 | 324 | 90 | 398 | 102 |
| 90 | 116 | 196 | 98 | 239 | 83 | 281 | 87 | 342 | 94 | 412 | 101 |
| 111 | 108 | 205 | 94 | 248 | 85 | 289 | 88 | 358 | 95 | 426 | 104 |
| 133 | 106 | 214 | 90 | 256 | 86 | 303 | 88 | 372 | 98 | 443 | 107 |
| 160 | 106 | 221 | 86 | 264 | 86 | 312 | 89 | 388 | 100 | 460 | 107 |
| | | | | | | | | | | 479 | 107 |

*Summary.*

The period of S Cassiopeiæ (Phillips's Group II.) was 614 days from J.D. 2422439 back to 2413832, and possibly to the earliest observations near J.D. 2401613, though its period *may* then have been 605 days, showing an increase from the early observations to the later.

## *Stellar Parallaxes Determined at the Royal Observatory, Greenwich.*

*(Communicated by the Astronomer-Royal.)*

The following list is a continuation of the results obtained with 26-inch telescope, and published annually in the November number of the *Monthly Notices* since 1920.

In the column Proper Motion the first value given is from meridian observations; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax, the relative proper motion has been used, *not* the meridian proper motion.

The Probable Error has been calculated by the formula

$$r = 0{\cdot}6745 \times \sqrt{\frac{[vv]}{m-\mu}} \times \frac{1}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion, and applies to both R.A. and Dec. when the latter is given.

Generally six comparison stars of mean magnitude 11–12 have been used, the magnitude of the parallax star being reduced to equality by means of the rotating sector.

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type. | R.A. Dec. 1900·0. | Proper Motion. R.A. | Proper Motion. Dec. | Parallax. | P.E. | No. of Plates. |
|---|---|---|---|---|---|---|---|---|---|
| | | m | | h m s | ″ | ″ | ″ | ″ | |
| 1 | O.A. 17 | 8·2 | | 0 3 39 | +·207 | +·021 | +·015 | ±·005 | 24 |
| | 65° 3 | 9·2 | | +65° 54′ | +·187 | | | ±·005 | |
| 2 | O.A. 388 | 8·2 | K0 | 0 23 53 | +·107 | +·013 | +·002 | ±·008 | 22 |
| | 74° 14 | 8·7 | | +74° 41′ | +·090 | | | ±·008 | |
| 3 | C. 84 | 9·5 | | 0 35 33 | −·250 | −·051 | +·017 | ±·008 | 19 |
| | 82° 16 | 11·1 | | +82° 45′ | −·275 | | | ±·008 | |
| 4 | O.A. 776 | 8·2 | G5 | 0 44 22 | +·171 | +·002 | −·014 | ±·008 | 18 |
| | 77° 29 | 8·8 | | +78° 5′ | +·134 | | | ±·002 | |
| 5 | L. 1721 | 8·1 | G5 | 0 56 16 | +·237 | −·154 | +·036 | ±·010 | 19 |
| | 68° 67 | 8·2 | | +68° 42′ | +·209 | | | ±·010 | |
| 6 | C. 179 | 9·3 | | 1 11 2 | +·218 | +·024 | −·012 | ±·007 | 24 |
| | 83° 28 | 9·7 | | +83° 17′ | +·173 | | | ±·006 | |
| 7 | C. 202 | 9·1 | | 1 22 51 | +·300 | +·152 | +·020 | ±·007 | 21 |
| | 82° 38 | 9·5 | | +82° 20′ | +·279 | | | ±·007 | |
| 8 | 76° 69 | 9·3 | | 1 59 44 | +·196 | −·094 | +·001 | ±·008 | 23 |
| | | 9·6 | | +77° 6′ | +·120 | | | ±·009 | |

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type. | R.A. Dec. 1900·0. | Proper Motion. R.A. | Proper Motion. Dec. | Parallax. | P.E. | No. of Plates. |
|---|---|---|---|---|---|---|---|---|---|
| | | m | | h m s | ″ | ″ | ″ | ″ | |
| 9 | O.A. 4198 | 9·4 | | 3 46 20 | +·178 | −·136 | +·001 | ±·005 | 28 |
| | 69° 230 | 10·0 | | +69° 33′ | +·196 | −·159 | | ±·002 | |
| 10 | Gr. 762 | 7·8 | F8 | 3 54 58 | +·183 | −·293 | +·018 | ±·007 | 20 |
| | 74° 184 | 7·4 | | +74° 54′ | +·173 | −·290 | | ±·002 | |
| 11 | 67° 330 | 9·5 | | 4 18 29 | +·168 | −·183 | +·003 | ±·007 | 25 |
| | | 10·8 | | +67° 38′ | +·176 | | | ±·006 | |
| 12 | K. 948 | 9·5 | | 5 30 18 | +·078 | −·157 | +·017 | ±·006 | 20 |
| | | 10·7 | | +75° 39′ | +·061 | | | ±·007 | |
| 13 | 75° 255 | 9·5 | | 6 19 47 | −·027 | −·191 | +·019 | ±·006 | 25 |
| | | 10·2 | | +75° 37′ | −·084 | | | ±·006 | |
| 14 | A.C. 74° 2509 | .. | | 6 29 7 | ·000 | −·122 | +·003 | ±·007 | 24 |
| | | 10·3 | | +74° 34′ | +·001 | | | ±·007 | |
| 15 | L. 13924 | 8·5 | K0 | 7 10 52 | −·074 | +·103 | +·031 | ±·005 | 21 |
| | 67° 483 | 9·0 | | +67° 50′ | −·076 | | | ±·006 | |
| 16 | 72° 421 | 9·5 | | 8 22 27 | −·079 | −·211 | +·028 | ±·005 | 15 |
| | | 11·0 | | +71° 54′ | −·064 | | | ±·007 | |
| 17 | O.A. 9258 | 8·4 | K0 | 8 39 7 | +·045 | −·104 | +·015 | ±·005 | 27 |
| | 67° 559 | 9·2 | | +67° 49′ | +·031 | | | ±·006 | |
| 18 | O.A. 11171 | 9·0 | | 10 46 48 | −·425 | +·168 | +·041 | ±·006 | 26 |
| | 76° 404 | 9·6 | | +76° 36′ | −·433 | | | ±·002 | |
| 19 | P. XIII. 96 | 6·5 | G5 | 13 22 35 | −·389 | +·215 | +·023 | ±·007 | 24 |
| | 64° 949 | 7·0 | | +63° 46′ | −·404 | | | ±·006 | |
| 20 | O.A. 14410 | 8·0 | K0 | 14 10 15 | +·161 | +·009 | +·030 | ±·010 | 19 |
| | 68° 771 | 9·0 | | +68° 3′ | +·153 | | | ±·009 | |
| 21 | O.A. 14843 | 8·9 | K0 | 14 40 49 | −·112 | −·011 | +·020 | ±·008 | 28 |
| | 69° 765 | 9·2 | | +69° 32′ | −·094 | | | ±·005 | |
| 22 | O.A. 14901 | 8·9 | K0 | 14 46 12 | −·118 | +·012 | +·017 | ±·007 | 27 |
| | 68° 801 | 9·6 | | +68° 23′ | −·087 | | | ±·004 | |
| 23 | L.F. 2603 | 7·0 | K2 | 15 5 40 | −·101 | +·043 | −·003 | ±·008 | 21 |
| | 72° 670 | 8·4 | | +72° 16′ | −·060 | | | ±·007 | |
| 24 | 74° 618 | 9·2 | | 15 26 28 | −·029 | +·236 | +·027 | ±·010 | 29 |
| | | 10·1 | | +74° 50′ | −·020 | | | ±·006 | |
| 25 | 67° 910 | 9·5 | | 15 38 19 | −·360 | +·143 | +·008 | ±·008 | 23 |
| | | 10·0 | | +67° 8′ | −·331 | | | ±·006 | |

| No. | Name and B.D. No. | Mag. Vis. Photog. | Spectral Type. | R.A. Dec. 1900·0. | Proper Motion. R.A. | Proper Motion. Dec. | Parallax. | P.E. | No. of Plates. |
|---|---|---|---|---|---|---|---|---|---|
| | | m | | h m s | ″ | ″ | ″ | ″ | |
| 26 | 65° 1092 | 8·8 | G5 | 15 56 43 | +·162 | −·176 | +·039 | ±·008 | 28 |
| | | 9·4 | | +65° 41′ | +·191 | | | ±·007 | |
| 27 | L. 33553 | 7·3 | K0 | 18 5 13 | −·070 | +·102 | +·022 | ±·009 | 29 |
| | 65° 1241 | 8·0 | | +65° 4′ | −·059 | | | ±·006 | |
| 28 | 69° 968 | 8·9 | K2 | 18 10 49 | −·015 | +·155 | +·029 | ±·013 | 22 |
| | | 9·1 | | +69° 38′ | −·040 | | | ±·008 | |
| 30 | $\pi$ Draco | 5·3 | A2 | 19 20 10 | +·019 | +·041 | +·002 | ±·013 | 24 |
| | 65° 1345 | 4·7 | | +65° 31′ | +·028 | | | ±·010 | |
| 31 | 68° 1345 | 8·3 | | 22 55 9 | +·567 | +·278 | −·004 | ±·008 | 25 |
| | | 9·0 | | +68° 29′ | +·570 | | | ±·006 | |
| 32 | P. XXIII. 218 | 6·3 | K2 | 23 47 32 | +·328 | +·052 | +·103 | ±·008 | 19 |
| | 74° 1047 | 7·1 | | +74° 59′ | +·309 | +·054 | | ±·002 | |

No. 30. A neutral-tinted filter employed.

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## *Le nombre des solutions qui se présentent dans la détermination d'une orbite elliptique.* By J. F. Cox.

(*Communicated by Professor H. C. Plummer.*)

Dans les " Meddelanden fran Lunds Astronomiska Observatorium," série II, No. 7 (Lund, 1911), M. Charlier a étudié l'emploi de l'équation de Lagrange dans le problème de la détermination des orbites, et a concrétisé ses résultats dans un graphique qui est devenu classique.

M. Plummer, dans son " Introductory Treatise on Dynamical Astronomy" (Cambridge, 1918), a étudié la représentation géométrique de M. Charlier et a énoncé quelques propositions nouvelles. Si l'on fait usage de coordonnées bipolaires et que l'on choisisse comme pôles les points E et S, représentatifs de la Terre et du Soleil, en appelant $\rho$ la distance à la Terre et $r$ la distance au Soleil, l'équation

$$\rho = \frac{1}{\kappa}\left(1 - \frac{1}{r^3}\right), \qquad (1)$$

dans laquelle $\kappa$ est un paramètre, représente une famille de courbes, qui ont été déterminées par points pour les valeurs de $\kappa$ qui sont admissibles.

Le graphique comporte en outre 3 courbes spécialement intéressantes:

Le cercle de centre S et de rayon 1, qui correspond à $\kappa = 0$;

La " courbe limite " d'équation

$$1-\frac{1}{r^3}+\frac{3}{2}(1+\rho^2-r^2)=0; \quad . \quad . \quad . \quad (2)$$

La " courbe singulière " d'équation

$$3\rho^2=2r^5-5r^2+3.$$

La " courbe singulière " est le lieu des points de contact des tangentes menées par E à chacune des courbes $\kappa$.

M. Stroobant, Professeur à l'Université de Bruxelles, m'a conseillé l'étude des travaux de MM. Charlier et Plummer, et croit utile de me faire signaler la propriété suivante : *La " courbe limite " est le lieu des points où les tangentes en E aux courbes $\kappa$ rencontrent celles-ci une seconde fois.*

En effet, dans le système de coordonnées choisi, l'équation générale d'une tangente est

$$\frac{\cos\alpha}{\cos\beta}=\frac{d\rho}{dr}$$

en appelant $\alpha$ et $\beta$ les angles qu'elle fait avec les rayons vecteurs issus de E et de S.

En tenant compte des conditions relatives à E ($r=1$, $\alpha=180°$) on obtient pour les courbes $\kappa$

$$\cos\beta=\frac{\kappa}{3}. \quad . \quad . \quad . \quad . \quad (3)$$

L'élimination de $\kappa$ entre (1) et (3) permet de retrouver l'équation (2) si l'on tient compte de l'identité

$$r^2\equiv 1+2\rho\cos\beta+\rho^2.$$